Abstracted Quantitative Structures: Using Quantitative Reasoning to Define Concept Construction

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Kevin C. Moore, Biyao Liang, Irma E. Stevens, Halil I. Tasova, and Teo Paoletti

Steffe and Thompson enacted and sustained research programs that have characterized students' (and teachers') mathematical development in terms of their conceiving and reasoning about measurable or countable attributes (see Steffe & Olive, 2010; Thompson & Carlson, 2017). Thompson (1990, 2011) formalized such reasoning into a system of mental operations he termed *quantitative reasoning*, and researchers have since adopted quantitative reasoning to characterize individuals' meanings within topical and related reasoning areas. For instance, researchers have adopted a quantitative reasoning perspective to explore how individuals construct or understand exponential relationships (Castillo-Garsow, 2010; Ellis et al., 2015), graphs or coordinate systems (Frank, 2017; Lee, 2017; Lee et al., 2019), and trigonometric functions (Moore, 2014; Thompson et al., 2007). Relatedly, researchers have adopted quantitative reasoning to explore individuals' meanings for concepts including rate of change (Byerley & Thompson, 2017; Johnson, 2015a, 2015b), function (Oehrtman et al.,

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2008; Paoletti & Moore, 2018), and accumulation (Thompson & Silverman, 2007), to name a few. More generally, researchers have related quantitative reasoning to other types of reasoning processes including multiplicative reasoning (Hackenberg, 2010; Tzur, 2004), algebraic reasoning (Ellis et al., 2020; Smith III & Thompson, 2007), generalization (Ellis, 2007), problem solving (Carlson et al., 2003), and transfer (Lobato & Siebert, 2002). An additional example that we leverage significantly in this chapter is reasoning about quantities changing in tandem, or *covarying* (e.g., Carlson et al., 2002; Johnson, 2012, 2015b; Saldanha & Thompson, 1998; Stalvey & Vidakovic, 2015).

The aforementioned studies and their authors' research agendas vary in the extent they focus on local or longitudinal development. Carlson et al. (2002) conducted a set of clinical interviews with calculus students in order to provide a localized description of their covariational reasoning. In contrast, Steffe and Olive (2010) provide a longitudinal model for children's construction and development of fractional schemes and concepts. Both grain sizes are critical to mathematics education research, with localized studies forming the foundation for longitudinal and generalized descriptions of individuals' cognition. But, each grain size has an associated cost. Localized studies can make it difficult for a researcher to characterize an individual's abstraction of a concept due to the extensive work necessary to make claims about an individual's in-the-moment reasoning. Longitudinal studies can make it difficult for the researcher to incorporate nuanced discussions of individual's in-the-moment construction of a concept due to the focus on described operations and their development in ways independent of particular contexts.

In this chapter, we introduce the construct *abstracted quantitative structure* (AQS) to marry the two aforementioned grain sizes and enable sensitivity to both localized and longitudinal development. Defined generally, an AQS is a system of quantitative operations a person has interiorized to the extent they can operate *as if* it is independent of specific figurative material.[1] That person can thus re-present this structure in several ways, and an AQS enables an individual to accommodate to novel experiences permitting the associated quantitative operations. Importantly, the AQS construct provides researchers (and teachers) criteria for characterizing individuals' construction and abstraction of concepts, and the AQS criteria are generalizable across concepts rather than specific to a particular concept. Researchers conducting work in the area of quantitative reasoning have made significant progress in articulating its role in the learning of particular concepts, yet generalized descriptions of how concept construction can be framed in terms of quantitative reasoning are less detailed or prevalent. This lack of specificity likely limits researchers' abilities to apply quantitative reasoning to their work (Drimalla et al., 2020; Thompson, 2008). This is especially problematic due to the extent researchers referenced above have shown quantitative reasoning to be a key component to students constructing a mathematics that is generative, coherent, generalizable, and sophisticated. It is thus important that researchers have the tools necessary to clarify meanings associated with these reasoning processes. In addition to providing generalizable criteria

[1] As we elaborate below, no conceptual structure is truly representation free.

by which to describe concept construction, the AQS construct helps make more explicit the role of quantitative reasoning in students' learning and development of key mathematical concepts.

In this chapter, we first introduce the AQS construct and its criteria. Due to its criteria being informed by numerous perspectives and extant constructs, we then introduce background information that underpins the AQS construct. With the background information in place, we provide a more detailed discussion of the AQS construct and its criteria. As part of this discussion, we draw from data to illustrate both indications—those actions that are consistent with particular AQS criteria—and contraindications—those actions that are inconsistent with particular AQS criteria—of individuals having constructed an AQS.[2] Following our empirical examples, we discuss the AQS in terms of how it can inform both research and teaching.

1 Introducing the Abstracted Quantitative Structure Construct

von Glaserfeld (1982) defined a *concept* as, "any structure that has been abstracted from the process of experiential construction as recurrently usable…must be stable enough to be re-presented in the absence of perceptual 'input'" (p. 194). Our notion of an AQS applies and extends von Glasersfeld's definition of concept to the area of quantitative and covariational reasoning. In the introduction, we defined an AQS as a system of quantitative (including covariational) operations a person has interiorized to the extent he or she can operate as if it is independent of specific figurative material. Adapting and extending von Glaserfeld's definition, an AQS is a system of quantitative operations (or quantitative structure) that an individual has interiorized so that it:

(1) is recurrently usable beyond its initial experiential construction;
(2) can be re-presented in the absence of available figurative material including that in which it was initially constructed;
(3) can be transformed to accommodate to novel contexts permitting the associated quantitative operations;
(4) is anticipated as re-presentable in any figurative material that permits the associated quantitative operations.

Our development of the Criteria 1–4 (C1–C4) defining an AQS is informed by perspectives on quantitative reasoning and covariational reasoning, distinctions between figurative and operative thought, and different forms of re-presentation. To provide the proper foundation for further defining and illustrating the criteria

[2] We underscore that we do not consider an AQS to be an exhaustive description of the meanings that can be associated with some concept. An AQS is a construct that can be used to describe a meaning for a concept that is rooted in quantitative and covariational reasoning, and we limit our discussion to such meanings except when contrasting them with alternative meanings to clarify AQS defining criteria.

associated with an AQS, we introduce critical definitions and perspectives in this section.

1.1 Quantitative Reasoning

Thompson (2011) defined quantitative reasoning as the mental operations involved in conceiving a context as entailing measurable attributes (i.e., quantities) and relationships between those attributes (i.e., quantitative relationships). A premise of quantitative reasoning is that quantities and their relationships are idiosyncratic constructions that occur and develop over time (e.g., hours, weeks, or even years). A researcher or a teacher cannot take quantities or their relationships as a given when working with students or teachers (Izsák, 2003; Moore, 2013; Thompson, 2011). Furthermore, and reflecting the criteria of an AQS presented below, a researcher or teacher should not assume a student has constructed a system of quantities and their relationships based on actions within only one context (e.g., situation, graph, or formula).

An important distinction is Thompson's (Smith III & Thompson, 2007; Thompson, 1990) use of *quantitative operation/magnitude* and *arithmetic operation/measure*. The former refers to the mental actions involved in constructing a quantity via a quantitative relationship. The latter are actions used to determine a quantity's numerical measure. Following Thompson (1990), we illustrate these distinctions using a comparison between two heights. Thompson (1990) described that an additive comparison requires one to construct an image of the measurable attribute that indicates by how much one height exceeds the other height (Fig. 1). Constructing such a quantity (i.e., a difference in heights) through the quantitative operation of comparing two other quantities additively does not depend on having specified measures. Returning to Fig. 1, no arithmetic operations are needed to conceive of a difference in heights as a measurable attribute, nor are they needed to conceive that difference as the black segment. *Arithmetic operations*, on the other hand, are those operations between specified or generalized measures such as addition, subtraction, or multiplication, that one uses to evaluate a quantity's measure. Such operations often occur in the context of symbols like inscriptions or glyphs. The arithmetic operations used to evaluate a quantity may reflect those quantitative operations that form the quantity (Fig. 2a, in which subtraction is used to calculate the difference quantity) or they may reflect other contextual relationships and properties (Fig. 2b, in which multiplication is used to calculate the difference quantity) (Thompson, 1990).

We provide Thompson's distinction between quantitative operations and arithmetic operations to emphasize that the *enactment* of quantitative operations occurs in the context of figurative material that permit those operations. As Steffe described in the context of the use of formulas versus graphs, "operations have to operate on something and that something is the figurative material contained in the operations, figurative material that has its origin in the construction of the operations" (L. P. Steffe, personal communication, July 24, 2019). To illustrate, consider Fig. 3. For

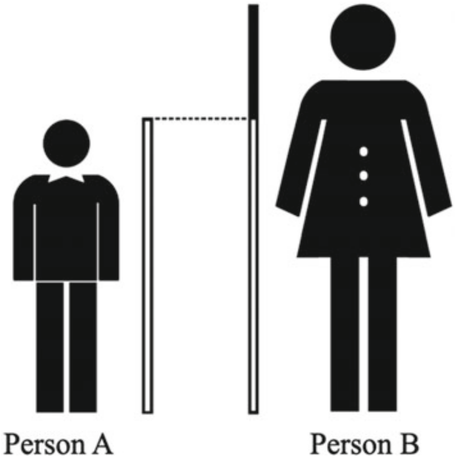

Fig. 1 An image of an additive comparison based in magnitudes

(a)

Height A: 45 inches
Height B: 60 inches

$$60 - 45 = 15$$

Height B exceeds Height A by 15 inches.

(b)

Height B is 4/3 times as large as Height A. Height A is 45 inches.

$$\frac{1}{3} \cdot 45 = 15$$

Height B exceeds Height A by 15 inches, because it exceeds Height A by 1/3 of Height A.

Fig. 2 The arithmetic operations that might be used to evaluate the measure of a quantity

example, the symbol "3" was socially negotiated as a way to signify those operations involved in measuring some magnitude as three of some unit. The symbol "3" is not designed to afford the enactment of quantitative operations.[3] On the other hand, a segment (more naturally) provides figurative material to assimilate via quantitative operations as having a measure of "3". Figure 3 illustrates a sequence that involves operations associated with units coordination including creating a unit and iterating that unit to determine the segment is some number of times as large as the unit (Steffe & Olive, 2010).

Addressing this distinction's implications for the AQS criteria, because symbols including glyphs and inscriptions are typically not used for the purpose of providing the figurative material to operate on quantitatively, students operating with symbols as such provides limited evidence of them enacting quantitative operations (Liang & Moore, 2021; Moore, Stevens, et al., 2019a, 2019b; Van Engen, 1949). On the other

[3] We acknowledge that we can identify creative ways to partition the symbol, but it is not used for such purposes. A stronger example of the distinction between a symbol and figurative material that permits quantitative operations is provided in the following section.

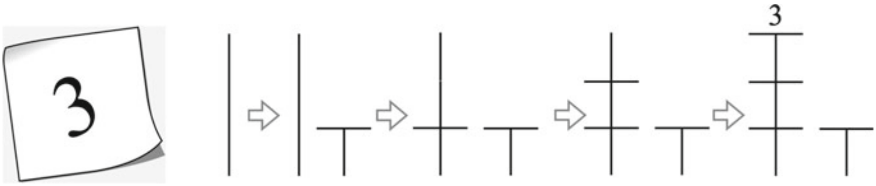

Fig. 3 The symbol "3" and a segment that affords enacting the mental operations signified by "3"

hand, coordinate systems, displayed graphs, phenomena, physical objects (e.g., a Ferris wheel or fraction blocks), and the like provide figurative material in which quantities can to be conceived and quantitative operations can be constructed and enacted. Thus, a student's activity with them can provide evidence of that student's engagement in reasoning quantitatively at that moment. It is for that reason that our defining the AQS criteria is with reference to contexts like coordinate systems, phenomena, and physical objects. We again illustrate the relevance of our focus in the following section and in the context of covarying quantities (Fig. 5) rather than quantities in a fixed state (Fig. 1 or Fig. 3).

1.2 Covariational Reasoning

A form of quantitative reasoning is covariational reasoning, which is defined as the actions involved in constructing relationships between two quantities that vary in tandem (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Researchers have identified that covariational reasoning is critical for key concepts of K–16 mathematics including function (Carlson, 1998; Oehrtman et al., 2008), modeling dynamic situations (Carlson et al., 2002; Johnson, 2012, 2015b; Paoletti & Moore, 2017), and calculus (Johnson, 2015a; Thompson & Silverman, 2007; Thompson, 1994b). Researchers have also illustrated that covariational reasoning is critical to constructing function classes (Ellis, 2007; Hohensee, 2014; Lobato & Siebert, 2002; Moore, 2014).

Carlson et al. (2002), Confrey and Smith (1995), Ellis and colleagues (Ellis, 2011; Ellis et al., 2020), Castillo-Garsow and colleagues (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013), Johnson (2015a, 2015b), and Thompson and Carlson (2017) have each detailed covariation frameworks and mental actions. Reflecting the emphasis of the empirical examples we use below, we narrow the present chapter's focus to *Mental Action 3* (Fig. 4, MA3) identified by Carlson et al. (2002). MA3 refers to coordinating and comparing quantities' amounts of change, which is a critical mental action to differentiating between nonlinear and linear growth (Paoletti & Vishnubhotla, this volume) and various function classes (Ellis et al., 2015; Moore, 2014). MA3 is also important for understanding and justifying that a graph and its curvature

Mental Action	Descriptions of Mental Actions
MA1	Coordinating the value of one variable with changes in the other
MA2	Coordinating direction of change of one variable with changes in the other variable
MA3	Coordinating amount of change of one variable with changes in the other variable
MA4	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable
MA5	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function

Fig. 4 Carlson et al., (2002, p. 357) covariational reasoning mental actions

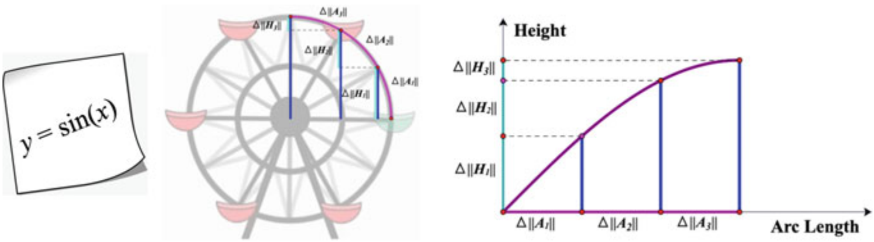

Fig. 5 For equal increases in arc length (colored in pink) from the 3 o'clock position, height (colored dark blue) increases by decreasing amounts (colored in light blue)

appropriately model covarying quantities of a situation (Fig. 5) (Stevens & Moore, 2016), and MA3 provides a foundation for rate of change reasoning (Johnson, 2015b; Thompson, 1994a). Furthermore, such reasoning enables understanding invariance among different representations of quantities' covariation (Liang & Moore, 2021; Moore et al., 2013; Norton, 2019), which is an AQS criterion we detail below.

Returning to the previous section's discussion of figurative material and the enactment of quantitative operations, Fig. 5 illustrates how a coordinate system and phenomenon provide material to assimilate via quantitative operations associated with MA3 and the sine relationship (Liang & Moore, 2021; Moore, 2014). The written "$y = \sin(x)$" was socially negotiated as a way to signify the operations involved in conceiving the sine relationship including the quantities' covariation. The symbol "$y = \sin(x)$" is not designed to afford the enactment of those quantitative operations, whereas the coordinate system and phenomenon provide contexts more organic to investigations of individuals' quantitative reasoning and, hence, defining the AQS criteria.

1.3 *Figurative and Operative Thought*

Although contexts like coordinate systems and phenomenon afford the enactment of quantitative operations, they also afford numerous other ways of reasoning, including those that might be incompatible with quantitative operations. We thus draw on Piagetian notions of *figurative* and *operative* thought (Piaget, 2001; Steffe, 1991), and particularly Thompson's (1985) extension of Piaget's distinction (see Moore, Stevens and et al., (2019b) for more), in our defining an AQS. The distinction between the two forms of thought enable us to differentiate between foregrounded aspects of thought including how those foregrounded aspects of thought may or may not support conceiving invariance among different actions and contexts. Defined generally, figurative thought is often based in and constrained to carrying out activity including physical actions, mental actions, motion, and imitations so that such activity produces a particular state. Operative thought is dominated by logico-mathematical operations, their re-presentation, and possibly their transformations. Operative thought foregrounds "intrinsic necessity, as opposed to successful solutions by chance or successful solutions that have simply been observed" (Piaget, 2001, p. 272), and available figurative material including that of the physical and perceptual kind is subordinate to the associated mental operations. Furthermore, meanings rooted in operative thought enable the reproduction or imagining of unavailable figurative material such that said material is a consequence of the construction of those mental actions. Quantitative and covariational reasoning are examples of operative thought due to their basis in logico-mathematical operations (Steffe & Olive, 2010; Thompson, 1994b).

To illustrate the figurative and operative distinction, Steffe (1991) characterized a child's counting scheme as figurative if his counting *required* re-presenting particular sensorimotor actions and operative if it entailed unitized records of counting that did not require the child to re-present particular perceptual material or sensorimotor experience. Relevant to the present chapter, Moore, Stevens and et al., (2019b) illustrated figurative graphing meanings in which prospective secondary teachers' graphing actions were constrained to particular figurative features (e.g., drawing a graph solely left-to-right) even when they perceived those features as constraining their ability to graph a relationship. In contrast, Moore, Stevens and et al., (2019b) described that a prospective secondary teacher's graphing meaning is operative when mental operations associated with quantitative and covariational operations persistently dominate perceptual and sensorimotor features of their graphing actions. Such a meaning enables an individual to graph a relationship across different coordinate orientations and coordinate systems and come to understand those graphs as quantitatively equivalent despite their perceptual differences or differences that occur in the sensorimotor experience of drawing a graph (Fig. 6). A student's construction of such a meaning illustrates Thompson's (1985) emphasis on the distinction of figurative and operative thought as an issue of "figure to ground" (p. 195), in which that which is operative on one level can become figurative on another level as it becomes the source material for subsequent operations and transformations.

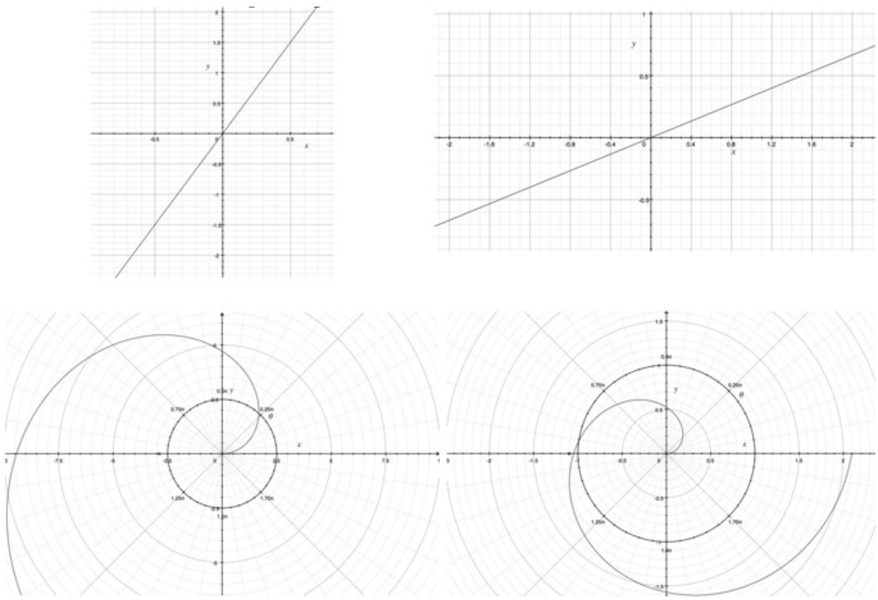

Fig. 6 Four graphs that differ perceptually and each represent one varying quantity being three times as large as another varying quantity (clockwise from top-left: $y = 3x$, $x = 3y$, $r = 3\theta$, $\theta = 3r$)

1.4 Three Forms of Re-presentation

Building on these aforementioned scholars' work on students' figurative and operative thought, Liang and Moore (2021) further operationalized these constructs in terms of a critical feature of an individual's cognition—an individual's ability to *re-present* their thought. According to Piaget (2001), von Glasersfeld (1995), and other constructivist scholars (e.g., Hackenberg (2010) and Steffe and Olive (2010)), re-presentation refers to an individual's ability to bring forth an image of schemes and operations that were enacted previously. Building on these scholars' collective works, Liang and Moore conceptualized three manifestations of an individual's re-presentational activities. The first form of re-presentation requires an individual to mentally generate some substitute for *all* sensory material that was present in prior experience but is absent currently (von Glasersfeld, 1995). For example, an individual can re-present MA3 in the context of a blank sheet of paper by recalling a Ferris wheel (or circle) and partitions (drawn or imagined) that correspond to amounts of change in two quantities.

The second form of re-presentation is similar to the first form, but it allows for the presence or supply of minimal figurative material or stimuli whose reconstructions are trivial to an individual. Using the same example as above, we can offer the individual a drawn circle after they've experienced a Ferris wheel animation and ask them to reconstruct MA3 in such context. This form of re-presentation is less strict than the first form, because it allows for some figurative material, and particularly that of the perceptual kind (e.g., the drawn circle), to be made available to an individual by a researcher.

The third form of re-presentation involves an individual *transforming* and regenerating operations enacted from a prior experience to accommodate a novel context. For example, after an individual constructs MA3 in the Ferris wheel (or circle) context, they can recall those operations and then transform and regenerate those operations under the constraints of the Cartesian coordinate system's quantitative organization to produce a graph representing the equivalent covariational relationship (see the graph given in Fig. 5). We use the word transform because this latter construction requires the individual to modify the operations associated with figurative material from the circle (e.g., incremental arcs and vertical segments) into material that differs perceptually due to the orthogonal orientation on the coordinate system. We underscore that this third form of re-presentation is different from the previous two forms in that it requires a transformation to occur in order to accommodate a novel context in a way that preserves some form of mathematical equivalence, and it thus forms an apropos example of operative thought. Collectively, these three forms of re-representation enable us to draw distinctions between each of the AQS criteria.

2 Further Defining and Illustrating the Abstracted Quantitative Structure Criteria

Recall that the criteria for an AQS is a system of quantitative operations that an individual has interiorized so that it:

(C1) is recurrently usable beyond its initial experiential construction;
(C2) can be re-presented in the absence of available figurative material including that in which it was initially constructed;
(C3) can be transformed to accommodate to novel contexts permitting the associated quantitative operations;
(C4) is anticipated as re-presentable in any figurative material that permits the associated quantitative operations.

C1 and C2 are consistent with the first two forms of re-presentation and associated examples discussed in the prior section. With respect to C2, it involves an individual having constructed a quantitative structure that is re-presentable in thought, and the individual can regenerate the operations with respect to those contexts experienced previously. It does not require that the individual be able to transform those operations to accommodate or generate a novel context.

Clarifying C3, a feature of an AQS is that it can accommodate novel contexts through additional processes of experiential construction with figurative material of which such construction has not previously occurred. We use *accommodation* to refer to when an individual modifies or reorganizes their meanings in order to establish a state of equilibrium or understanding (Montangero & Maurice-Naville, 1997; von Glasersfeld, 1995). Some forms of accommodation can be quite significant and require a fundamental change in an individual's ways of operating (Steffe & Olive, 2010), while other forms of accommodation can be more subtle and involve a way of operating being used with respect to novel sensory material (i.e., generalizing assimilation as defined by Steffe and Thompson (2000)). To illustrate, and building off the example provided in the previous section, after an individual has constructed some quantitative structure in the Cartesian coordinate system, they might recall those operations and then transform and regenerate them under the constraints of the Polar coordinate system's quantitative organization to produce a graph representing the equivalent covariational relationship (see the graphs given in Fig. 6). Such an accommodation is consistent with the third form of re-presentation discussed in the prior section, and it is a hallmark of operative thought because it entails an individual transforming and using operations of their quantitative structure to accommodate to novel quantities and associated figurative material, as opposed to having fragments of figurative activity dominate their thought (Thompson, 1985).

Whereas C3 refers to the enactment (and transformation) of quantitative operations in specified contexts, C4 refers to an individual anticipating the mathematical properties (e.g., quantities' covariation) of the quantitative operations constituting an AQS independent of any particular instantiation of them. The individual understands the operations and associated properties as not constrained to any particular quantities and figurative material. It is in this way that the quantitative operations of an AQS are abstract; the individual not only understands that the operations are re-presentable in previous experiences, but they also anticipate that the operations and their properties *could be* relevant to novel but not yet had experiences. Or, similarly, the individual anticipates that the operations and their properties *could be* relevant to experiences so complex in their figurative material or specified quantities that the individual does not yet have the fluency to enact those operations.

C4 extends beyond the forms of re-presentation discussed in the previous section due to it involving the anticipation of a hypothetical experience that has not been previously experienced. As an example, after graphing some relationship in numerous coordinate systems and orientations (e.g., the Cartesian and polar coordinate systems), a student may anticipate that there likely exist coordinate systems not yet experienced such that those coordinate systems enable enacting and representing the AQS and its mathematical properties; the student understands that they will need to adjust their operations to the specific quantitative constraints of the yet-to-be-experienced coordinate system, while also anticipating that the properties of those operations will remain the same (i.e., a linear relationship entails a constant rate of change no matter the coordinate system it is graphed within). As an alternative example, an individual could experience a novel or complex pair of quantities in which they have difficulty or cannot enact particular quantitative operations. Despite that difficulty, C4 involves the individual being able to anticipate the mathematical properties of a particular AQS. For instance, a student might not yet have constructed the capacity to enact quantitative operations with a quantity like surface area, but the student could anticipate a linear relationship between the painted surface area and height of a sphere to mean the painted surface area of a spherical cap increases at a constant rate of change with respect to the painted height of a sphere (Fig. 7).

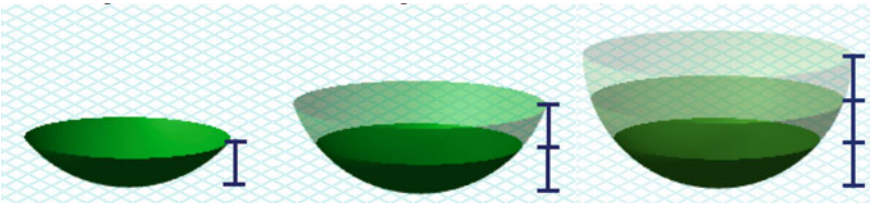

Fig. 7 For equal increases in height (marked on the right of the figure), the surface area of the spherical cap (colored in green) increases by equal amounts (each shaded band of surface area)

The student could then leverage that inference to explore the relationship between surface area and height without having to enact quantitative operations in the context of the sphere (Stevens, 2019).

2.1 Empirical Illustrations

The value of a construct aimed at explaining cognition rests in its ability to provide explanatory or descriptive accounts of individual activity. We use selected empirical examples in this section to illustrate the criteria of an AQS. Each example is drawn from a study that used clinical interview (Ginsburg, 1997) or teaching experiment (Steffe & Thompson, 2000) methodologies to build second-order models of student thinking (Ulrich et al., 2014). It was in our reflecting on second-order models that we identified themes in student reasoning, and the AQS construct provided a consistent way to frame data from those studies.

We acknowledge the way we have defined AQS presents inherent problems in attempts to characterize a student as having or having not constructed such. First, it is impossible to investigate a student's reasoning in every context in which an AQS could be relevant. This limits the strength of claims relative to C3. Second, to characterize a students' quantitative reasoning necessarily involves focusing on their enactment of operations in the context of particular figurative material. This limits the strength of claims relative to C3 and C4, and particularly attempts to characterize the extent an individual's reasoning is not constrained to specific figurative material or quantities. No conceptual structure is truly representation free, as "operations have to operate on something" (L. P. Steffe, personal communication, July 24, 2019), but a conceptual structure can be abstracted to the extent the individual can symbolize and project it as if it is representation free. For these above reasons, we find it productive to discuss a student's actions in terms of *indications* and *contraindications* of their having constructed an AQS per the defined criteria, as opposed to claiming whether a student *has* or *has not* constructed an AQS. Indications are those actions that are consistent with particular AQS criteria, and contraindications are those actions that are inconsistent with particular AQS criteria.

In what follows, we provide examples that illustrate indications and contraindications of C2 and C3. C1 is trivial in its illustrations and is indicated by an individual re-enacting the associated quantitative operations to assimilate some experience separate from but identical in form to the original enactment of those operations. A contraindication of C1 is an individual enacting associated quantitative operations only in-activity, with each enactment of them being effortful and somewhat anew even when presented with what an observer considers the same figurative material as considered in previous activity (e.g., an identical task and/or animation). As we illustrate below, C4 is also somewhat trivial in its illustration, as it typically involves an individual making a verbal statement acknowledging the possibility of re-presenting particular relationships in situations not yet experienced. The illustration in Fig. 7 was one example of a potential anticipated linear relationship in a novel context. Another example is after numerous experiences graphing some relationship in multiple coordinate orientations and systems, the individual might acknowledge the possibility that other coordinate systems exist that entail a quantitative organization that affords re-presenting the operations associated with that relationship. Yet another example, although more complex in its form, is an individual thinking of hypothetical experiences they may work to occasion with a learner in order to support the learner's developmental of an AQS.[4]

2.1.1 A Contraindication of Re-Presentation in the Absence of Figurative Material (C2)

A critical criterion of an AQS is the ability to re-present that structure in the absence of available figurative material (and, often, in the context of transforming its operations to accommodate to novel contexts, i.e., C3). As a contraindication of re-presentation, consider Lydia's actions during a teaching experiment focused on trigonometric relationships and re-presentation (Liang & Moore, 2021). Prior to the actions presented here, Lydia had constructed incremental changes compatible with those displayed with the Ferris wheel in Fig. 5 to conclude that the vertical segment (Fig. 8, in green) increases by decreasing amounts (circled in Fig. 8c) for equal changes of arc length (i.e., MA3). We took her actions to indicate her reasoning quantitatively, particularly as she was able to reproduce her actions repeatedly in the context of the Ferris wheel animation (i.e., C1). We subsequently presented her the *Which One?* task (Fig. 9, also see [https://youtu.be/2pVVGl8eEr0]).

This task included a simplified version of a Ferris wheel (the left side of Fig. 9) with the position of a rider indicated by a dynamic point. The topmost blue bar (the right side of Fig. 9) displayed the arc length the rider had traveled counterclockwise from the 3 o'clock position. Lydia could vary the length of this bar by dragging its endpoint or by clicking the "Vary" button, and the dynamic point on the circle moved correspondingly. We asked Lydia to determine which of the six red bars, if any,

[4] We thank a reviewer for pointing out the relationship between an educator seeking to engender learning and their having constructed an AQS.

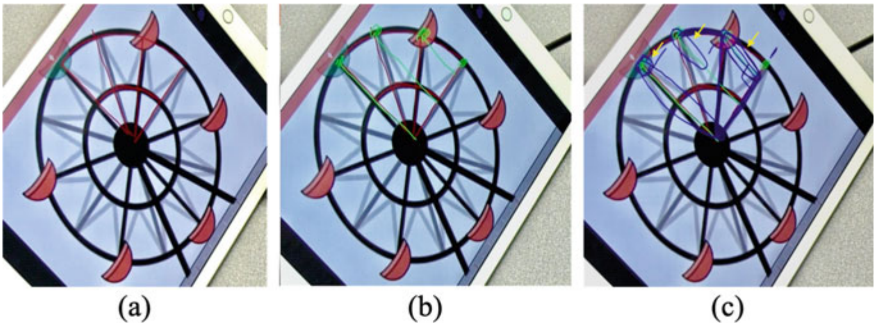

Fig. 8 Lydia's prior actions and their results (Liang & Moore, 2021, p. 303)

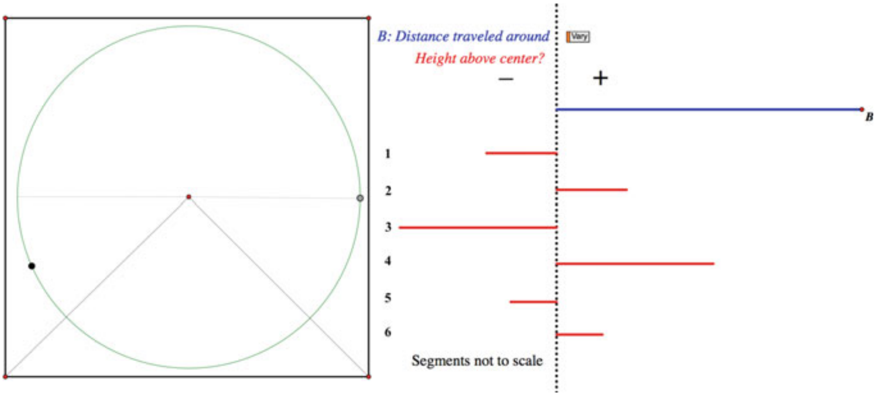

Fig. 9 The *Which One?* Task as presented to the student (Liang & Moore, 2021, p. 300)

accurately display the rider's height above the horizontal diameter as the rider's arc length varied (i.e., the sine relationship). The vertical dotted line provided a reference mark for the red bars, with a red bar emanating left being a negative magnitude and a red bar emanating right being a positive magnitude. The red bars were "free-moving" in that they could be repositioned and reoriented in the plane, including being reoriented and placed as a vertical segment emanating from the center of the circle. Our overall design was intended to determine the extent she could re-present her previous actions in a similar context with less figurative material immediately available than before (i.e., the Ferris wheel and its features, like the spokes in Fig. 8), but with novel material that might support her in enacting those operations (i.e., the red and blue segments oriented horizontally). For reference, the topmost bar is a normative solution, and the other bars vary with either different directional variation (e.g., positive or negative; decreasing or increasing) or different rates (e.g., constant, increasing, or decreasing rate) than the normative solution (see https://youtu.be/2pVVGl8eEr0).

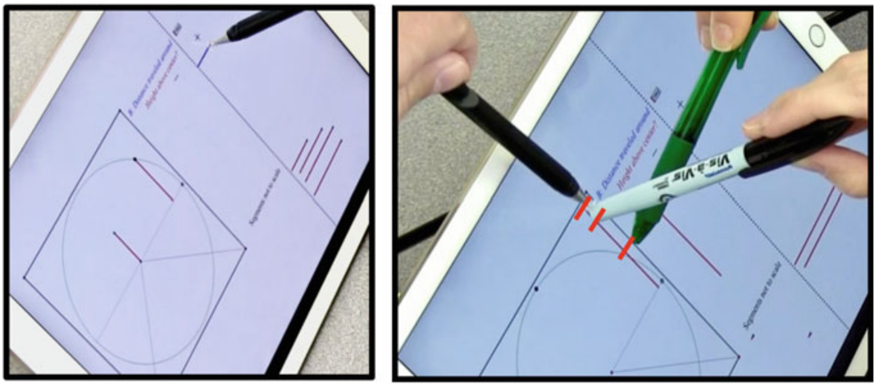

Fig. 10 Lydia re-orienting and checking the red segment that is the normative solution (left) and Lydia, with assistance, constructing partitioning activity (right, with the red partitions added to aid the reader) (Liang & Moore, 2021, p. 304)

As detailed in Liang and Moore (2021), Lydia became perturbed as to whether or not the horizontal red segment should vary at a changing rate with respect to the horizontal blue segment despite originally claiming that it should be based on her previous actions with the Ferris wheel (see identified changes circled in blue in Fig. 8). After much effort, she abandoned considering the segments in the horizontal orientation and re-oriented them vertically on the screen. Importantly, she persistently held in mind that her activity with the Ferris wheel led to the relationship of decreasing increases in height for successive equal changes in arc length (e.g., Fig. 8), but stated that she was unsure how to show such a relationship with the present segments. She eventually chose the correct segment solution by checking whether the heights matched at multiple static states within the displayed circle (Fig. 10, left).

At this point in the task, Lydia's activity had us question the extent she had constructed an AQS during her activity for the Ferris wheel. Specifically, she required re-orienting the red segments rather than either mentally rotating them vertically or leaving them horizontal and comparing their behavior with her image from the prior activity (i.e., a contraindication of C3). Furthermore, when re-oriented, she explicitly acknowledged having difficulty re-presenting the relationship from the Ferris wheel (i.e., a contraindication of C2). In an attempt to provide additional insights into her reasoning, after Lydia had chosen the normative solution, the teacher-researcher (TR) returned her to the question of whether the chosen red segment and blue segment entailed the same relationship as she identified in her previous activity (see Fig. 5, left):

Lydia: Not really…Um, I don't know. [*laughs*] Because that was just like something that I had seen for the first time, so I don't know if that will like show in every other case…Well, for a theory to hold true, it like – it needs to be true in other occasions, um, unless defined to one occasion.

TR.: So is what we're looking at right now different than what we were looking at with the Ferris wheel?

Lydia: No. It's – No…Because I saw what I saw, and I saw that difference in the Ferris wheel, but I don't see it here, and so –

TR.: And by you "don't see it here," you mean you don't see it in that red segment?

Lydia: Yes.

(Liang & Moore, 2021, p. 303).

In the present interaction, and as the interaction continued, Lydia expressed uncertainty as to how to determine if the blue segment and her chosen red segment entailed the same relationship she had illustrated in her previous activity, although she knew the segments were correct in static states. We underscore that Lydia held in mind the relationship she conceived in the Ferris wheel situation, and in the Ferris wheel illustration she could regenerate the operations as suggested by her work in Fig. 5 (i.e., an indication of C1). Consistent with a contraindication of C2, one possible explanation for her activity is that the Ferris wheel situation provided arms that supported her partitioning activity, and her activity at this time was reliant on the availability of that material (Liang & Moore, 2021). As a further contraindication of C2, it was only after the teacher-researcher introduced perceptual material using their pens (Fig. 10, right) that she conceived the red and blue segments' covariation as compatible with the MA3 relationship she had constructed in the Ferris wheel situation.

2.1.2 An Indication of Re-Presentation in the Absence of Figurative Material (C2)

As suggested by Lydia's activity, an indication of C2 would have been her regenerating amounts of change using the circle, the re-oriented red bars, and the blue bars, and without the assistance of the research team. As a more detailed indication of C2, consider Caleb's activity (Liang et al., 2018; Tasova et al., 2019) when engaging in the *Changing Bars Task* (Fig. 11). For the task, the red segment on the circle represents the magnitude of the point's height above the horizontal diameter and the blue segment represents the magnitude of the point's arc length from the 3 o'clock position (i.e., the sine relationship). The user was able to move the endpoint along the circle between the 3:00 position to the 12:00 position. On each displayed orthogonal pair, the user was able to drag the endpoint of the *red segment* in order to increase or decrease its magnitude. We asked Caleb to choose which, if any, of the orthogonal pairs—red representing height and blue representing arc length—accurately represents the relationship between the point's height and arc length as it moves a quarter of a rotation around the circle. Two of the pairs accurately represented the relationship.

Before Caleb engaged with the *Changing Bars Task,* he had engaged with numerous Ferris wheel and segment orientation tasks including *Which One?* His activity on those tasks indicated he had constructed a quantitative structure consistent with C1. We designed the *Changing Bars Task* to gain insights into the sophistication of his quantitative structuring. The *Changing Bars Task* thus includes subtle figurative differences including less figurative material than that of a Ferris wheel

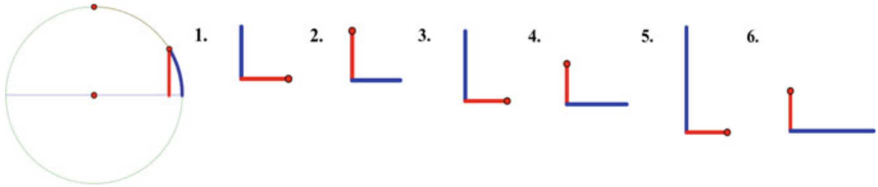

Fig. 11 *Changing Bars Task* (numbering of the six pairs provided for readers)

situation, pairs of orthogonally oriented bars (Fig. 11) rather than six height bars and one arc length bar (as in Fig. 9), and the ability to vary the red (height) segment rather than the blue (arc length) segment.

Summarizing Caleb's activity, he initially compared the amounts of change in arc length and amounts of change in height as the dynamic point traveled a small distance from the 3:00 position. He stated that, "…at the very beginning, … the height above the center and the distance traveled from 3:00 position should be similar." Caleb then repeated his actions near the 12:00 position, adding:

> *Caleb*: ...from this point [*pointing to the point denoted in orange in* Fig. 12b*]* ... to this point *[pointing to 12:00 position in* Fig. 12b], the height barely changes [*green segment in* Figs. 12b *and c (i.e.,* $\Delta\|H_3\|$*)*], but you're still traveling a fair distance around the circle [*blue annotation in* Fig. 12b *and blue segment (i.e.,* $\Delta\|A_3\|$*) in* Fig. 12c].

Caleb's actions suggest he constructed a gross additive comparison of $\Delta\|H_1\|$ with $\Delta\|A_1\|$ near the 3:00 position (i.e., $\Delta\|H_1\|$ is almost equal to $\Delta\|A_1\|$) and of $\Delta\|H_3\|$ with $\Delta\|A_3\|$ near the 12:00 position (i.e., $\Delta\|H_3\|$ is smaller than $\Delta\|A_3\|$). He also generalized this relationship across all cases in the first quarter of rotation. He stated, "the further you move away from the 3:00 position, the more variance there would be between the red (i.e., $\Delta\|H\|$) and the blue lines (i.e., $\Delta\|A\|$)." In this case, by "variance" he meant that $\Delta\|A\|$ became much bigger than $\Delta\|H\|$ as the dynamic point approached the 12:00 position, whereas an alternative meaning would be comparing $\Delta\|H\|$ magnitudes for successive changes in arc length. In other words, he was coordinating how the two quantities' changes compared to each other rather

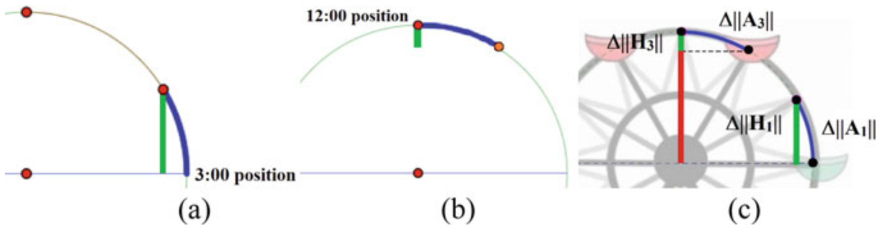

Fig. 12 Recreation of Caleb's activity in the *Changing Bars Task*. We introduce the Δ and magnitude notation to use in the narrative and highlight that Caleb's reasoning foregrounded magnitudes rather than (directed) measures

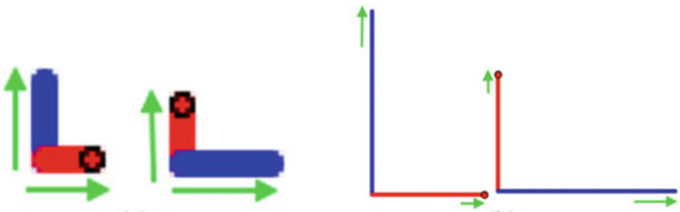

Fig. 13 Caleb's choice of two pairs of bars at **a** the beginning state and **b** the final state. The direction and length of each arrow indicates the direction and magnitude of change respectively

than fixing changes in one quantity and comparing the changes in the other quantity (Liang et al., 2018; Tasova et al., 2019).

Turning his attention to the orthogonal pairs, he dragged the end point of the red bar for a small amount from a start of near zero magnitude, and observed by how much the blue bar changed (Fig. 13a); he also dragged the end point of the red bar for a small amount towards the maximum length of both bars to observe by how much the blue bar changed (Fig. 13b). He moved aside all pairs which either the blue bar did not change by "almost equal" as the red bar near their minimal amounts or the blue bar did not change by noticeably larger amounts than the red bar near their maximum amounts. He selected two pairs (the normative solutions), the relevant behavior of which are described in the Fig. 13 caption.

In contrast to Lydia's activity above, Caleb's activity is an indication of C2 because the entirety of his solution suggests he did not need aspects of the Ferris wheel (e.g., the spokes) or denoted partitions on the segments to provide markers for his activity. Rather, he was able to mentally imagine particular actions and their results in the absence of available figurative material including that with which he had previously acted. Furthermore, he was able to conceive equivalence in his actions among both contexts without having to reflect upon and enact operations on produced figurative material.[5]

2.1.3 A Contraindication of Accommodation (C3)

As a contraindication of C3, we turn to Patty's activity when prompted to graph a covariational relationship in a different Cartesian orientation than she had previously graphed (Moore et al., 2019b). Patty was working the *Going Around Gainesville* (*GAG*) task (Fig. 14). Patty constructed a normative solution to *Part I* in ways that suggested her reasoning covariationally and re-presenting that relationship using a Cartesian graph (Fig. 15).

[5] We note that we do not consider Caleb's activity as an indication of C3 as he had previously experienced both circle and segment contexts repeatedly during the teaching experiment. His engagement suggested they were not novel relative to his perceived goal and activity.

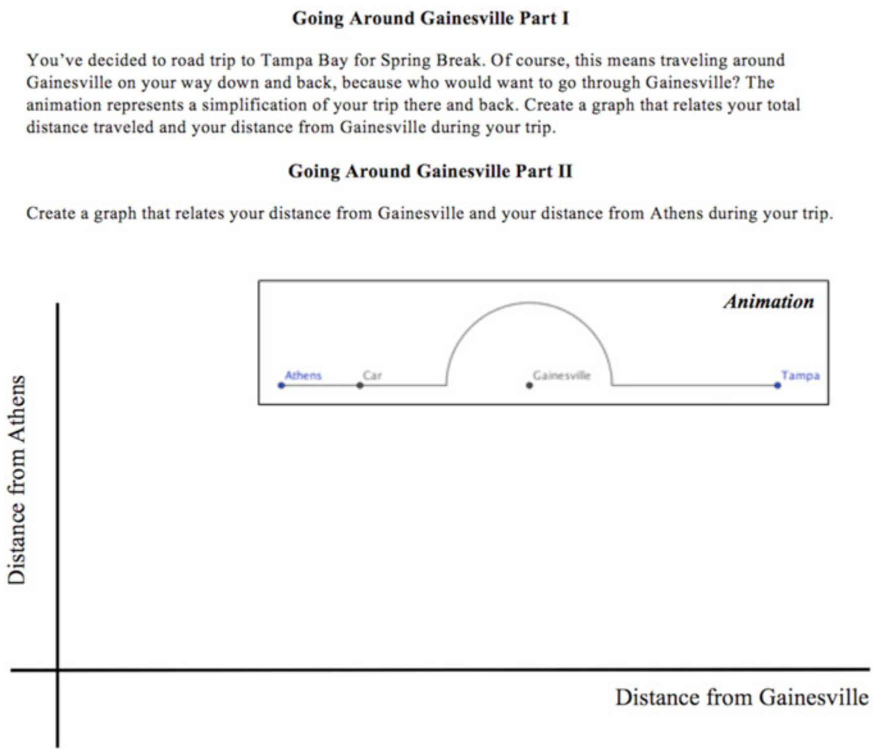

Fig. 14 The *Going around Gainesville* (*GAG*) task (Moore et al., 2019b, p. 4)[6]

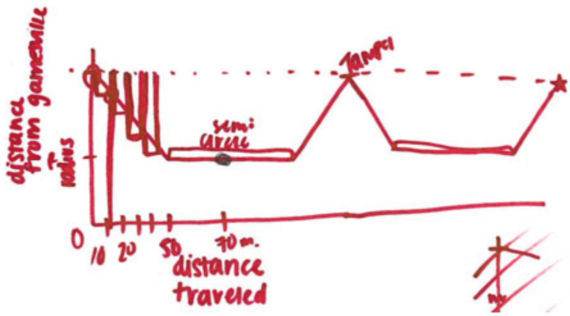

Fig. 15 Patty's work on *GAG* part I (Moore et al., 2019b, p. 13)

Moore et al. (2019b) provided a detailed account of Patty's solution to *GAG Part II*, which illustrated her experiencing a sustained, conscious perturbation that left her unable to complete the task to her satisfaction. We note that Patty anticipated graphing the same relationship she had previously constructed and graphed (i.e., a

[6] This task is a modification of the task Saldanha and Thompson (1998) presented.

quantity's magnitude decreasing as the other magnitude increased), but she encountered an irreconcilable perturbation when attempting to do so in the *Part II* coordinate orientation. Specifically, Patty had determined an initial point along the vertical axis and then motioned as if drawing a segment sloping downward left to right from this point. She later crossed out that point on vertical axis, as seen in the top-left of Fig. 16. She explained (see Fig. 16 for her work):

> *Patty*: I wanted to start here because I wanted to show that the distance was decreasing [motioning down and to the right from the point plotted on the vertical axis], but that means your distance from Athens is decreasing [tracing vertical axis from the initial point to the origin]…[turning her attention to the relationship from the animation] But your distance from Athens is growing. But your distance from Gainesville is decreasing. So, if that's growing and that's decreasing, so [draws an arrow pointing downward beside horizontal axis label and then an arrow pointing upwards beside the vertical axis label]
>
> [Patty then works for six additional minutes maintaining her 'starting' point on the vertical axis, without making progress, and explaining "this is so hard". She repeatedly identifies the distance from Gainesville as decreasing and the distance from Athens as increasing, including drawing a graph in an alternative axes orientation (i.e., Distance from Athens ("dA") being on the horizontal axis, see the bottom right of Fig. 16). She eventually has an insight.]
>
> *Patty*: Ohhhh, what if I started it like here [*plots point on the right end of the horizontal axis*]. Okay…but I don't want to start like, like I don't like starting graphs. You know I don't know work backwards that's weird…[*in the next minute and a half Patty draws in a normative initial segment of the graph, as seen in Fig. 16, hesitating throughout while explaining how the distances covary*] But it's backwards so I don't like it…My graph is from right-to-left, which is probably not right…[*describes the covariational relationship between the two distances*] I guess I just don't like this.
>
> *Int.*: And why don't you like it?
>
> *Patty*: Because it's backwards.
>
> *Int.*: And by backwards we mean?
>
> *Patty*: Backwards is traveling from right-to-left. But I think my graph is just, I think I'm just not clicking. I think I'm missing something.
>
> (Moore et al., 2019b, p. 14).

Recall that C3 involves transforming a system of quantitative or covariational operations in order to accommodate novel contexts permitting the associated quantitative operations. On one hand, Patty's activity is an indication of C3; she was able to transform MA3 operations enacted in the situation—a car traveling along a road—to construct a Cartesian graph representing the same amounts of change relationship (Fig. 15). On the other hand, Patty's activity is a contraindication of C3; she was unable to accommodate those operations enacted in the situation and previous graph to construct a Cartesian graph in a different quantitative orientation that she maintained as a correct representation of the relationship. Notably, figurative features of her graphing activity (e.g., "work[ing] backward," and "traveling right-to-left) constrained her ability to enact and sustain quantitative operations in the alternative Cartesian orientation. Patty's activity illustrates the complexity of C3, and we return to this complexity in the closing discussion.

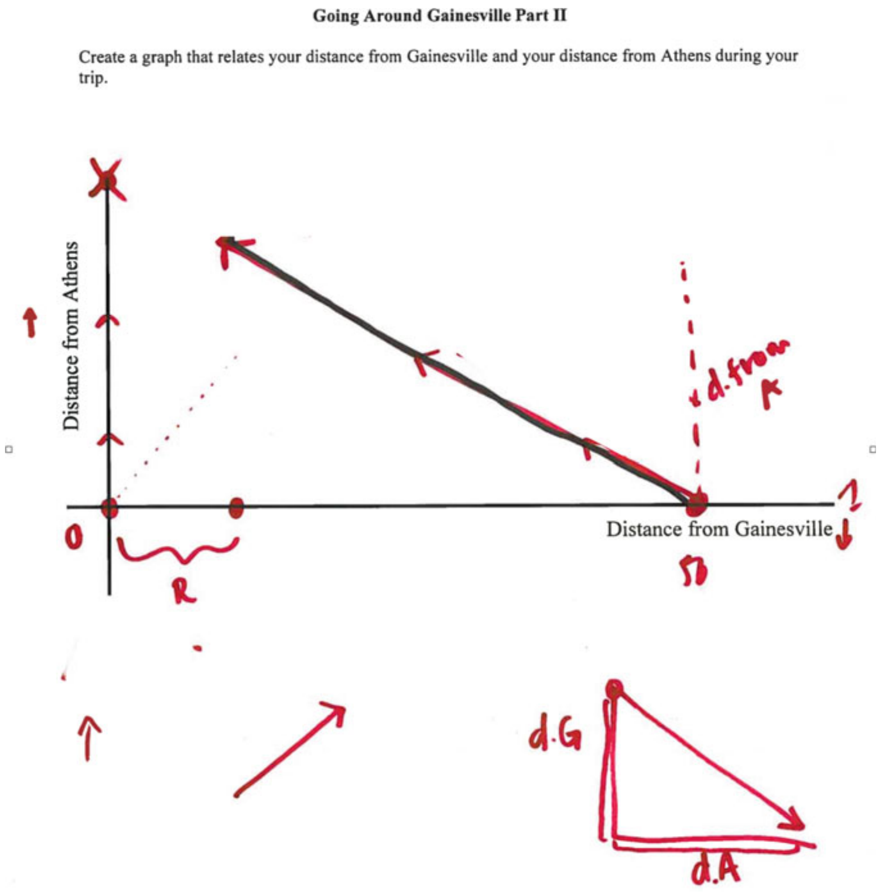

Fig. 16 Patty's graph for the first portion of the trip for *GAG* part II

2.1.4 An Indication of Re-presentation and Accommodation (C3)

As an indication of re-presentation and accommodation (i.e., C3), we turn to two prospective secondary teachers'—Kate and Jack—actions when asked to determine a formula for an unnamed polar coordinate system graph (Fig. 17, which is $r = \sin(\theta)$; see Moore et al. (2013) for the detailed study). After investigating a few points, Kate and Jack conjectured that $r = \sin(\theta)$ is the appropriate formula and drew from memory a Cartesian sine graph to compare to the polar graph. Important to note, Kate and Jack were not familiar with graphing the sine relationship in the polar coordinate system.

> *Kate*: This gets us from zero to right here is zero again [*tracing along Cartesian horizontal axis from 0 to* π]. So, we start here [*pointing to the pole in the polar coordinate system*].

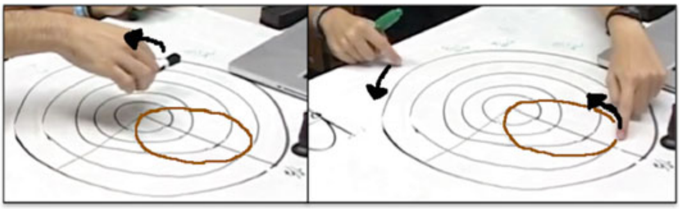

Fig. 17 Kate and Jack covary quantities with respect to the given graph (from Moore et al., 2013, p. 467)

> *Jack*: Yeah, and you're sweeping around because [*making circular motion with pen*], theta's increasing, distance from the origin increases and then decreases again [*Jack traces along Cartesian graph from 0 to* π *as Kate traces along corresponding part of the polar graph*].
>
> *TR.*: OK, so you're saying as theta increases the distance from the origin does what?
>
> *Jack*: It increases until pi over 2 [*Kate traces along polar graph*] and then it starts decreasing [*Kate traces along polar graph as Jack traces along Cartesian graph*].
>
> *TR.*: And then what happens from like pi to two pi.
>
> *Kate*: It's the same.
>
> *Jack*: Um, same idea except your, the radius is going to be negative, so it gets more in the negative direction of the angle we're sweeping out [*using marker to sweep out a ray from* π *to* $3\pi/2$ *radians – see* Fig. 17] until three pi over two where it's negative one away and then it gets closer to zero [*continuing to rotate marker*].
>
> *TR.*: OK, so from three pi over two to two pi, can you show me where on this graph [*pointing to polar graph*] we would start from and end at?
>
> *Kate*: This is the biggest in magnitude, so it's the furthest away [*placing a finger on a ray defining* $3\pi/2$ *and a finger at* $(1, \pi/2)$], and then [*the distance from the pole*] gets smaller in magnitude [*simultaneously tracing one index finger along an arc from* $3\pi/2$ *to* 2π *and the other index finger along the graph – see* Fig. 17].
>
> (Moore et al., 2013, p. 468).

Kate and Jack's actions indicate their having constructed (or constructing) a covariational relationship associated with the sine relationship such that they could take that relationship as a given in the Cartesian coordinate system. Furthermore, their actions indicate their transforming the associated operations to accommodate to a polar coordinate system displayed graph. Their activity enabled them to conceive two graphs as representing equivalent quantitative structures despite their perceptual differences, which is an indication of C3.

2.1.5 An Implication of Anticipation (C4) and Accommodation (C3)

In Kate and Jack's case, an indication of C4 would involve their identifying the *potential* of not yet experienced coordinate systems that enable re-presenting the same quantitative structure. In our experience it is difficult to gain evidence of C4, as it relies almost entirely on verbal descriptions of anticipation that stem from researcher prompting. Thus, rather than using this section to illustrate an indication of C4, we focus on an important implication of an individual's (whether they are a researcher, teacher, or student) actions that suggest their constructing a structure consistent with C4: their AQS is generative in their consideration of other individuals' work. By generative, we mean that their AQS is productive for assimilating a broad range of experiences with others in ways that are sensitive to the ways of operating of those others (Liang, 2021).

In our previous work, we documented prospective teachers' (PSTs') difficulties with attributing and valuing meanings rooted in quantitative and covariational reasoning to non-normative student work and coordinate system orientations, such as a student graphing x and y on the Cartesian vertical and horizontal axes, respectively (Lee et al., 2019; Moore et al., 2014, 2019a, 2019b). Attributing and valuing meanings rooted in quantitative and covariational reasoning to non-normative work (e.g., graphing under alternative axes orientations) necessitates the construction of an AQS. More specifically, an individual having constructed an AQS consistent with C1 through C4 positions the individual to anticipate some other person producing representations that, while novel to the individual, are viable (i.e., C4). Furthermore, the individual is positioned to accommodate to that person's produced representations via the transformation and regeneration of particular quantitative operations (i.e., C3).

As an illustration, we draw on Annika's activity as reported in Moore et al., (2019b). Annika was addressing a task (Fig. 18) presenting student work. Prior to this task, Annika's activity had indicated her anticipating rate of change as a coordination of quantities' variation that she could enact in any coordinate system or orientation so as long as she adjusted to the quantitative organization of that system and its orientation. Moore et al., (2019b) thus provided a task to see how she would attribute meaning to student work that was not clearly labeled but could be determined as viable in a number of ways.

As an indication of C4, Annika's immediate action was to consider the presented graph as a potentially viable graph of $y = 3x$, and she sought to determine coordinate orientations that enabled her to regenerate the quantitative operations she associated with a graph of $y = 3x$. Namely, she sought to determine labeling so that y varied by a magnitude 3 times as large as any corresponding variation in x. As potential labels, she identified positive x- and y-values oriented down and right of the origin, respectively, and she identified positive x- and y-values oriented up and left of the origin, respectively.

Following this interaction, and in an attempt to determine the extent Annika considered the graph as a viable representation of $y = 3x$, the teacher-researcher posed that a different student claimed that the line has "a negative slope" and thus is not a graph of $y = 3x$. She responded with the following, using positive x- and y-values oriented down and to the right of the origin (see Fig. 19), respectively, as her specific example:

> *Annika*: You'd have to notice that even though it looks like a negative slope [*making a hand motion down and to the right*] because we call it slope because it's visual and it's easy to visualize a negative and positive slope [*making hand motions to indicate different slopes*].

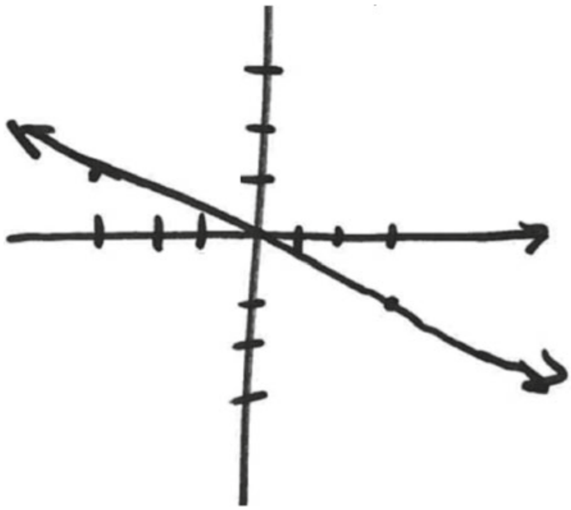

Fig. 18 The task presented to Annika, which was posed as a student solution to graphing $y = 3x$ (Moore et al., 2019b, p. 7)

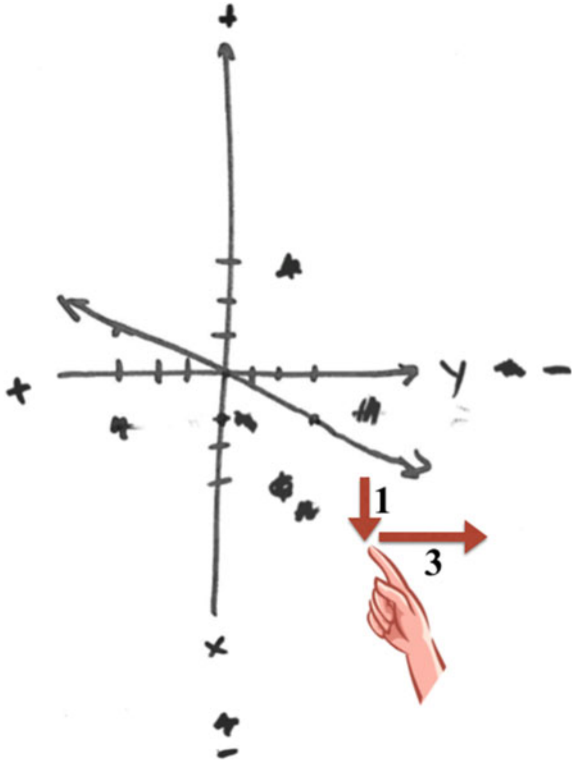

Fig. 19 Annika's annotated version of Fig. 18, each tick mark based on a unit change of 1 (Moore et al., 2019b, p. 13)

> But that's only visual on our conventions of how we set it up. Um, but like [*pointing to the graph*] if slope is rate of change, we can still see that for like equal increases of x [*making hand motions to indicate equal magnitude increases*] we have an equal increase of y [*making hand motions to indicate equal magnitude increases*] of three. And so for equal positive increase of one [*sweeping fingers vertically downward to indicate an increase of one*], we have an equal positive increase of three [*sweeping fingers horizontally rightward to indicate an increase of three*]. And so, it is a positive slope.
>
> (Moore et al., 2019b, p. 12).

Annika's response suggests her differentiating visual notions of slope from rate of change in a way that indicates her having constructed rate of change as a relationship between quantities' values as they vary. This relationship was not tied tightly to any particular coordinate system or orientation (i.e., C4). She then further enacted her scheme for constant rate of change to make sense of a graph in a particular orientation by re-presenting the quantitative operations she associated with a constant rate of change of 3 (i.e., changes in one value are a multiple of changes in the other value). Collectively, Annika's activity illustrates the powerful implications of an individual, and in this case a PST, having constructed an AQS in the context of their considering non-normative and unclear student work.

2.1.6 A Few Comments on Context

For consistency, we provided examples drawn from one population (prospective teachers) with a focus on particular mathematical ideas (e.g., covariation and graphing). However, our framing of an AQS is applicable to all age ranges and across numerous mathematical, scientific, and day-to-day domains (e.g. Steffe and colleagues work addressing students' counting and fraction schemes). Namely, we see the AQS criteria as relevant to topics and contexts including the construction of individual quantities (e.g., length), the combining of quantities to form other structures (e.g., coordinate systems or multiplicative objects), the construction of particular covariational relationships (e.g., the sine relationship), or the construction of a phenomenon itself (e.g., a Ferris wheel). For instance, an individual's construction of length as a quantity can start out tied to the experience of movement. Over time the individual may then construct height as a measurable attribute of a person, and then length as a more generalized measurable attribute of any number of concrete objects. Finally, that individual may conceive length as a measurable attribute they can impose on any span of space or object they come across in future experience. To illustrate further, we draw on the Faucet Task (Paoletti, 2019; Paoletti et al., accepted) to provide an example that spans students' mathematical understandings, understandings of systems, and lived experiences.

In the Faucet task, students interact with a dynamic applet that allows them to turn hot and cold knobs, with such turns resulting in changing amounts of water and temperature of water leaving the faucet (Fig. 20). Overall, the middle school students (aged 10–13) develop a quantitative structure that supports them in making additive (e.g., turning a knob on results in an increase in water) and ratio (e.g., turning the hot

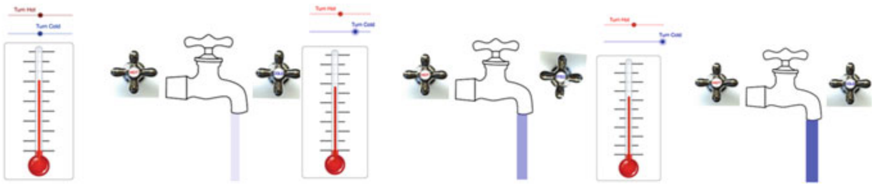

Fig. 20 Several screenshots of the *faucet task* with the cold-water knob being turned on from an initial state

knob on results in increasing the hot water relative to the cold water, thereby making the water hotter) comparisons to determine how temperature and amount of water covary. As an initial stage of the task, and with the aid of the applet, students develop an understanding of how the faucet system operates. For instance, if only the cold knob is turned, all the water comes out at the constant temperature of groundwater. Or, if only the hot knob is turned, all the water comes out a constant temperature set by the hot water heater. Likely due to their numerous experiences with faucets, middle school students often quickly develop a quantitative structure for the faucet system that satisfies C1–C2, thus being able to imagine changes in amounts of water and temperature of water in thought, as well as creating the relevant states and variations of the system using the applet. In other words, they can enact quantitative operations in order to control the faucet system as they choose.

The Faucet tasks illustrates that a conversation on the extent an individual has constructed an AQS can exist with respect to different contexts. For instance, while middle school students are able to construct a sophisticated quantitative structure with respect to the faucet system (i.e., C1–C2), it is often non-trivial for them to use a graph to re-present relationships constructed within the faucet system environment (i.e., C3). Similarly, Patty's difficulty graphing a relationship she conceived highlights this complexity. A student having developed a sophisticated situational quantitative structure that provides evidence of C1–C2 does not immediately imply the student will be able to transform or regenerate this structure in a new context like a coordinate system. As Patty's actions illustrate, it could be the case that while their situational quantitative structure is sophisticated, their meaning for the representational system in which they are attempting to re-present that structure might not afford such actions. Or, as Lydia's actions indicate, it could be the case that an individual's situational understanding is not sophisticated enough to either be re-presented in the absence of specific figurative material or be transformed and regenerated in a different representational system. We return to this point in the next section when discussing research and teaching implications.

3 Discussion and Implications

Throughout the results, we used students' activity to provide indications and contraindications of the four criteria for AQSs (C1–C4). We leveraged these criteria to highlight the different forms of re-presentation and to distinguish between students' meanings in terms of their foregrounding figurative material and activity and their foregrounding logico-mathematical operations (i.e., quantitative operations). For example, Lydia's initial activity was illustrative of the importance of the first two manifestations of re-presentation, as she required some figurative material available to coordinate which segment length represented the rider's vertical distance. We underscore that Lydia did not encounter much difficulty once all necessary figurative material was available; she assimilated the segments and their variation to quantitative operations. Rather, Lydia struggled (and explicitly acknowledged said struggle) to accommodate the relationship she constructed in a way that she could re-present it with novel, and partially unavailable figurative material.

The complexities Lydia experienced further demonstrates the power of Kate, Jack, and Annika's reasoning. Kate and Jack's activity exemplifies students leveraging the third manifestation of re-presentation by transforming and regenerating operations enacted from a prior experience to accommodate a novel context, which is an indication of operative thought. For example, not only did they re-present a quantitative structure and regenerate that structure in a novel context, they also abstracted the associated operations such that they could identify the same relationship within a perceptually different representational system. In Annika's case, her activity underscores that the construction of an AQS better positions an individual to understand the reasoning of others as their meanings are more malleable in the presence of novel figurative material.

Because this chapter serves as an introduction of the AQS construct and criteria, we spend the remainder of this section discussing potential research and teaching implications. We envision these implications to provide avenues by which researchers and teachers can move the AQS construct forward. As we mentioned above, the value of a construct aimed at explaining cognition is measured by the extent it affords explanatory or descriptive accounts of individual activity, and such an affordance is best judged in the context of subsequent research and attempts to engender learning.

3.1 Research Implications

We find the criteria associated with an AQS to provide a grounding for researcher claims regarding students' (and teachers') quantitative and covariational reasoning in two primary ways. First, the AQS criteria provide a way to characterize the sophistication of a student's quantitative reasoning whether with respect to a phenomenon (e.g., a Ferris wheel or a faucet), a representational system (e.g., a coordinate system or number line), or a concept (e.g., rate of change or the sine relationship). Prior to developing the AQS criteria, our research team often found it difficult (and unproductive) to characterize a student as reasoning quantitatively or not. For instance, for students for which we only had gathered data indicating C1, we were unsure whether those students were or were not reasoning quantitatively. But, as we gathered indications (or contraindications) of C2 through C4 through the design of interactions that afforded such indications, we found that we were able to develop more nuanced models of the students' quantitative reasoning. Furthermore, as we gathered indications or contraindications of C2 through C4, we found that we had more evidence to make viable claims regarding students' quantitative reasoning across each context in which their reasoning occurred. Thus, we see the AQS criteria as providing guidance for researchers as it relates to building evidence and making claims regarding the affordances and constraints of a student's quantitative reasoning. The AQS criteria emphasize that a researcher's sensitivity to figurative and operative distinctions in an individual's thought should be based in the researcher's sustained interactions with the individual and iterative testing of hypotheses regarding the individual's re-presentation and regeneration capacities.

Second, and relatedly, the AQS construct and criteria emphasize the importance of situating models of students' quantitative reasoning, including the perturbations students experience when enacting such reasoning. By providing criteria that draw attention to recurrent usability, re-presentation of prior experience, regeneration within novel contexts, and the anticipation of regeneration in future experience, the AQS criteria enable researchers to situate their claims regarding students' quantitative reasoning by being explicit about both the quantitative operations under study, as well as the contexts and figurative material in which those operations can be enacted or anticipated. Actions like those of Patty and Lydia highlight that it is important for researchers to simultaneously attend to students' meanings for various phenomena (e.g., a faucet system or Ferris wheel ride), various representational systems (e.g., Cartesian coordinate system and polar coordinate system), and the quantitative relationships they construct within a phenomenon or representational system. For instance, Patty's experienced perturbation did not stem from the relationship she constructed within the phenomenon. It instead stemmed from her meaning for graphing in the Cartesian coordinate system. The AQS criteria draw attention to this distinction with a focus on quantitative operations, their enactment, and their regeneration, and this distinction can prove powerful when designing for other interactions with a student as we illustrate in the next section.

With respect to students' mathematical development, we acknowledge to this point we have not explicitly defined *abstraction* in this chapter. This is notable given Piaget's (2001) extensive use of different forms of abstraction. It is not an oversight that we have not explicitly defined abstraction to this point, but rather an indication that we have not yet operationalized the construct of an AQS in terms of its construction and development. We are in the process of conducting and designing additional studies to provide insights into the construction and development of such structures as it relates to particular relationships, topics like rate of change, and representational systems. We hypothesize that the construction of AQSs occurs through cyclical processes of pseudo-empirical, reflecting, and reflected abstraction, in which what becomes operative and conscious at one level becomes the figurative ground for further processes of abstraction. We also note that our studies to date suggest the developmental interdependence of particular structures. Particularly, a student having constructed an AQS for a particular relationship can support them in constructing an AQS for a representational system as they attempt to re-present the relationship in that system. One can imagine that Patty's perturbation could have led to a powerful accommodation to her Cartesian graphing meanings had the setting afforded particular instructional interventions. Similarly, a student is afforded the opportunity to construct an AQS associated with a topic like rate of change through repeated opportunities of constructing AQSs of particular relationships that entail different rates of change as those relationships can become a source of reflection and abstraction.

Lastly, providing a set of criteria for a construct invites the question whether said criteria have a hierarchy. There is a natural hierarchy with C2 through C4, as they move from re-presenting a previous experience, to accommodating to a novel context, and ultimately anticipating hypothetical future contexts. With that said, we have not conducted the empirical work necessary to articulate developmental stages or shifts, and thus hesitate to make claims relative to the hierarchy of the criteria. We do note that C2 and C3 are each a subset of C1, as both C2 and C3 require the enactment of the associated operations beyond their initial experiential construction. For instance, a student reenacting MA3 by reproducing a Ferris wheel (or circle) on a (provided or imagined) blank sheet of paper and producing (via drawing or imagining) partitions to construct amounts of change is an example of C2 and necessarily implies their constructed quantitative structure is recurrently usable beyond its initial experiential construction. C1 is broader than C2 and C3, and particularly C2, because it includes cases that do not require regenerating figurative/perceptual material, including that in which it was initially constructed. For example, a researcher might give a student their finished work from a previous experience, and the student could assimilate it with little effort in order to recall their previous actions and their results.

3.2 Teaching Implications

With respect to teaching, and mirroring several of the research implications, the AQS criteria provide a lens for instructional design, both with respect to curricula and a teacher's or student's interactions with students. With respect to curricula, our experience (at least in the US) leads us to believe a majority of mathematics curricula do not intentionally provide students opportunities to reason in ways consistent with C2 through C4 (Moore et al., 2013, 2014, 2019a, 2019b). As a notable example, within US 6–12 curricula function classes are almost presented and graphed exclusively in the Cartesian coordinate system with their independent variable along an axis oriented horizontally and with positive values oriented to the right of the origin. Furthermore, the vertical axis is almost exclusively oriented with positive values above the origin.

This practice constrains students' opportunities to construct an AQS because it does not afford repeated occasions to transform and regenerate the quantitative operations associated with a function class to accommodate to novel contexts (i.e., other coordinate orientations or systems). Instead, this practice affords the propagation of meanings that foreground figurative aspects of thought and lower forms of abstraction (Moore et al., 2019b; Paoletti et al., 2018a, 2018b; Thompson, 2013; Thompson et al., 2017). More generally, it likely restricts students in constructing a coordinate system as a form of an AQS, as students only experience one form of a coordinate system.

That the majority of curricula do not intentionally target C2 through C4 is also problematic for the type of classroom interactions that occur in the context of that curricula. Echoing our comments about building evidence for making claims relative to students' quantitative reasoning, the products students are likely to produce are constrained (cf. diSessa et al., 1991). Ultimately, this limits the variety and multitude of student products teachers and fellow students have to consider, compare, and leverage. For students, this limits their opportunities to construct and compare different forms of reasoning and representations. For a teacher, this limits their ability to assess the extent their students have constructed a sophisticated and flexible meanings for the topic under consideration. In short, the aforementioned limitations in curricula have the likely consequence of restricting or devaluing the variety of student actions necessary to support reasoning in ways consistent with C2 through C4.

Responding to this issue, we have found promising results in designing instructional experiences that incorporate the criteria set forth in this chapter (Moore et al., 2014). Specifically, when designing for instruction, we have found it productive to start with answering the question: What is critical to a concept and what are merely conventional practices for representing that concept? By answering that question and differentiating between the two, an educator can identify the mental operations they want to foster and then use the AQS criteria to inform their instruction and interactions with students. As a concise example, consider the sine relationship. With respect to covariation and MA3, we consider a critical aspect of the sine relationship to be that for successive equal increases from 0 in one quantity's value, the other quantity's value increases by decreasing amounts, then decreases by increasing amounts, then decreases by decreasing amounts, and then increases by increasing amounts before repeating that pattern.[7] With that aspect in mind, targeting C2 and C3 then involves providing students a variety of experiences to construct, coordinate, re-present, and regenerate the corresponding mental operations. Such experiences could include having students determine how the arc length traveled by various Ferris wheel riders varies in relation to the riders' heights above the center of the Ferris wheel, and then extending that to any object traveling along a circle (i.e., C2). From there, the students could be tasked with regenerating those operations in different Cartesian

[7] We note that there are several other key aspects to understanding the sine relationship, including measuring quantities in radius lengths, proportionality, and periodicity (Bressoud, 2010; Moore, 2014; Thompson, 2008).

and polar coordinate orientations (i.e., C3). Ideally, when reflecting on that collection of experiences and identifying the invariant properties of their operations, the students would be positioned to anticipate the corresponding MA3 relationship as potentially relevant for future experiences in not-yet experienced coordinate systems and phenomenon (i.e., C4).

In providing the concise example in the prior paragraph, we do not imply that such a process is simple or quick. Rather, each step and construction along the way is quite effortful on the part of the learner, and we view each AQS criterium as drawing educators' attention (whether teachers, researchers, or curricular designers) to an area deserving intense focus. As an example, consider a student's construction of a coordinate system, which is itself a quantitative structure. Lee (2017) illustrated, and underscoring the emphasis of C1 and C2, a student's construction of a coordinate system is a complex coordination of mental operations that develop over time. Students need repeated opportunities to construct and coordinate the operations associated with a coordinate system if they are to use coordinate systems productively. Instructional design should thus be built around student opportunities to construct coordinate systems that at least satisfy C1 through C3. Additionally, other mental operations are necessary when constructing a graph within a coordinate system, particularly when that graph involves regenerating and representing a relationship from a different context like a Ferris wheel (Lee et al., 2018; Moore, 2021; Moore & Thompson, 2015; Paoletti et al., 2018a). Here, the AQS criteria C1 through C3 draw attention to the importance of a student not only constructing a sophisticated quantitative structure in a context like the Ferris wheel, but also to their being able to transform and regenerate that structure in the context of the coordinate system. Student actions as such cannot be taken for granted and should instead be explicit targets of instructional design.

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