Middle School Students' Meanings of Points from Quantitative and Covariational Reasoning Perspectives

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We examine the meanings students give to points when they are graphing relationships between quantities in dynamic, experiential contexts. Using data from teaching experiments with middle-grades students, we illustrate two main categories of meanings: iconic and quantitative. We then introduce four distinct subcategories of meanings: (a) iconic and transformed iconic translations (a point represents an object or location), (b) nonunited points (a point represents a single quantity's magnitude), (c) spatial-quantitative multiplicative objects (a point is an object or location with quantitative properties), and (d) quantitative multiplicative objects in conventional and nonconventional planes (a point represents two quantities' magnitudes). We discuss the implications of these meanings for research, teaching, and curriculum development.

Keywords: Graphs and graphing; Quantitative reasoning; Covariational reasoning; Middle school; Teaching experiment

Graphing is essential for understanding mathematical concepts (Thompson & Carlson, 2017). Beginning to work with graphs represents a critical moment in middle school mathematics, fostering powerful learning and supporting functional thinking and algebraic reasoning (Ellis, 2011; Van Dyke & White, 2004). However, students often face challenges in interpreting graphs that affect their learning of algebra and calculus ideas like function and rate of change (Clement, 1989; Hattikudur et al., 2012). They may view graphs as literal pictures of situations or motions rather than as representations of quantitative relationships (Clement, 1989; Johnson et al., 2020). Although these approaches may work in isolated instances, they are not flexible or generative for long-term learning and sense making in STEM contexts (Glazer, 2011; González, 2021; Panorkou & Germia, 2021; Rodriguez et al., 2019).

Research on graph interpretation and construction has historically addressed general graphing behaviors (e.g., Ponte, 1984; Wavering, 1989) or focused on specific mathematical concepts, such as slope and intercepts (e.g., Hattikudur et al., 2012). It has also taken on different perspectives, such as incorporating embodiment-based learning opportunities (e.g., Botzer & Yerushalmy, 2008; Nemirovsky et al., 1998; Radford et al., 2009). This perspective highlights how students use their kinesthetic experience to make sense of graphs by associating changes in motion (e.g., speed and direction) with a graph's shape (e.g., aspects of concavity and height; Deniz & Dulger, 2012; Nemirovsky et al., 1998). Alternatively, the multiple representations perspective (e.g., Fonger, 2019; Knuth, 2000; Nitsch et al., 2015) focuses on how students relate graphs to other forms of representation, such as tables and equations, emphasizing flexible movement among representations (Moschkovich et al., 1993; Yerushalmy & Schwartz, 1993). In this article, we promote a perspective that focuses on students' *quantitative and covariational reasoning*: how individuals construct and reason about measurable attributes that vary in magnitude (Thompson, 2011; Thompson & Carlson, 2017). Such a focus orients us to developing detailed models of students' meanings for the graphical objects they construct—such as points, curves, and axes—as they attempt to graph quantities and their relationships.

Researchers have suggested that quantitative and covariational reasoning form productive foundations for students' graphing meanings (e.g., Carlson et al., 2022; Frank, 2017; Johnson, 2022; Paoletti et al., 2023; Paoletti, Gantt, & Vishnubhotla, 2022; Thompson et al., 2017). Developing meanings using quantitative and covariational reasoning enhances students' and preservice teachers' graphing skills (e.g., Lee et al., 2019; Paoletti & Vishnubhotla, 2022; Tasova, 2021). A key theme within the research on individuals' quantitative and covariational reasoning is the importance of constructing *multiplicative objects*, which are objects that result from cognitively uniting two or more attributes (i.e., quantities' magnitudes) simultaneously (Frank, 2017; Saldanha & Thompson, 1998; Thompson & Carlson, 2017; Thompson et al., 2017). Another theme in this line of research is the importance of students conceiving of a graph as an emergent trace of how two united quantities' values vary simultaneously (*emergent shape thinking*; Moore, 2021; Moore & Thompson, 2015). This meaning affords a student the opportunity to make sense of a graph as a product of enacted mathematical operations,

instead of conceiving of a graph as a shape qua shape in which the student's meanings are restricted to learned facts associated with perceptual shape patterns and properties (*static shape thinking*; Moore, 2021; Moore & Thompson, 2015; Rodriguez & Jones, 2024). Reasoning about graphs using emergent shape thinking can serve as an important foundation for STEM disciplines because it allows for the modeling and understanding of dynamic, complex systems (Czocher et al., 2022; Paoletti, Lee, et al., 2022; Rodriguez et al., 2019).

To date, researchers, including those employing the perspective we adopt here, have taken several approaches to investigate students' meanings for graphs, coordinate systems, and points. Some researchers have explored the relationship between coordinate systems and the mental actions involved in organizing space (Lee et al., 2020). Others have examined students' meanings for points in relation to concepts such as functions or rate of change (e.g., Carlson et al., 2002; David et al., 2019; Paoletti & Moore, 2017; Ponte, 1984). One remaining gap in this area of research is an understanding of students' meanings of points in reference to dynamic situations and independent of specific concepts. Namely, the field lacks nuanced models explaining how students *re-present*—or reconstruct during a new experience (von Glasersfeld, 1995)—quantitative relationships when creating graphs to capture dynamic situations understood to entail those perceived quantitative relationships. As graphing activities with moving objects (e.g., Budak et al., 2018; Cory, 2010; Fernández, 2006) or dynamically changing objects (e.g., DePeau & Kalder, 2010; Koyunkaya et al., 2015; Singleton & Ellis, 2020) gain momentum in curricula, the need grows to systematically document students' diverse meanings of points, including the extent to which they conceive of points as representations of quantitative relationships and multiplicative objects.

In this article, we introduce models that clarify characteristics of students' meanings for points with respect to the attributes of a situation they intend to capture with their points and, hence, graphs. These models articulate what students conceive of their graphing activity to re-present and the extent to which their actions are quantitative when they create graphs that convey a dynamic, experiential situation and its quantitative relationships. We introduce these models as a framework for categorizing students' meanings of points on a plane, developed through a teaching experiment with four middle-grades students. Because the framework represents abstracted models of middle-grades students' thinking, it contributes novel insights into foundational graphing skills along two primary features of students' meanings: The framework distinguishes between meanings that are iconic or quantitative and between meanings according to the extent they entail conceiving of a point as a multiplicative object. As a whole, the framework contributes to researchers' pursuit of the answer to the important question: What are middle school students' meanings of points when they are engaged in graphing activities that emphasize quantitative and covariational reasoning within experiential contexts?

Theoretical Perspective

Our perspective on students' meanings is grounded in von Glasersfeld's (1995) radical constructivism, which views an individual's knowledge as idiosyncratic and fundamentally unknowable to others. This theory frames our study, but we also adopt Thompson and Harel's definition of meaning within their system of knowledge that is based on Piagetian notions of scheme and assimilation (Thompson, 2016; Thompson et al., 2014). In this system, schemes are cognitive structures of mental actions, images, and objects that individuals enact in relevant situations to achieve specific results (Thompson et al., 2014). Schemes are implicative, with more advanced ones incorporating greater complexity and actions (Dubinsky, 1991; Thompson et al., 2014). Unlike traditional definitions of schemes that emphasized a stimulus-action-result structure (von Glasersfeld, 1995), Thompson et al. (2014) defined *meaning* as a way to refer to an individual's schemes while emphasizing that a meaning is an ensemble of actions and related schemes.

Assimilation is the active, constructive process of using an existing cognitive structure to organize a current experience and establish a state of equilibrium, which corresponds to achieving an understanding (Piaget, 2001; von Glasersfeld, 1995). This involves assimilating an experience to a system of schemes and associated implications (Thompson et al., 2014). For example, a student could assimilate a point on a coordinate plane into their conception of a physical object on a map (e.g., a bike), anticipating that as the bike moves left or right, the point should move identically. Alternatively, a student could assimilate a point as the result of uniting two segments on coordinate axes, anticipating that these segments re-present two quantities' magnitudes from the situation. They may also anticipate plotting points to record covarying magnitudes, matching the segments' covariation with the situation's magnitudes.

Gaining understanding through assimilation can be effortful, requiring cognitive reorganization (i.e., learning). We focus on participants' enacted meanings that achieved equilibrium through assimilation, rather than on the cognitive reorganizations and problem-solving actions that led to their initial construction of those meanings. Additionally, the reported meanings do not represent a participant's complete understanding or capacity. Individuals hold various meanings for a concept, and which meaning they enact is stochastic from a researcher's perspective. Any claim regarding a participant's meaning is a model providing merely one meaning that viably explains their in-the-moment action.

Background and Literature Review

Saldanha and Thompson (1998) and Thompson (2011) highlighted two key aspects of students' quantitative and covariational reasoning: (a) conceiving of each quantity itself and its dynamic variation, and (b) conceptualizing a multiplicative object by coupling two quantities while maintaining the image of the dynamic situation. In this section, we first summarize research on key aspects of quantitative and covariational reasoning in graph construction and interpretation. Then, we discuss the history of the concept of a multiplicative object and examine how graphs and coordinate systems have previously been studied.

Quantitative and Covariational Reasoning

We use "quantity" to refer to the conceptual entity that an individual constructs as an attribute of an object that allows measurement—whether anticipated or realized—with an associated unit (Thompson, 2011). Thompson (1990) described *quantitative reasoning* as "the analysis of a situation into a quantitative structure," where a *quantitative structure* is "a network of quantities and quantitative relationships" (p. 12). Researchers have emphasized the importance of conceiving of situations with respect to quantities and their relationships when learning foundational mathematical concepts such as rate of change, quadratic and exponential relationships, and functions (Carlson et al., 2002; Castillo-Garsow, 2012; Confrey & Smith, 1994; Ellis, 2011; Ellis et al., 2016; Johnson, 2015; Thompson & Carlson, 2017). Providing students with opportunities to conceptualize quantities and their varying relationships in dynamic situations helps them construct meaningful graphs, because graphing becomes a generative re-presentation of their mathematical thinking (Ellis et al., 2018; Frank, 2017; Johnson et al., 2020; Lee et al., 2019; Paoletti et al., 2023; Thompson, 2011). This understanding is different from merely plotting points through iconic translation (i.e., conceiving of graphs as a picture of a situation; Clement, 1989; Monk, 1992), memorized actions such as "over and up," or reproducing memorized shape associations such as "faster" means "curves up."

Covariational reasoning involves envisioning how two quantities' values vary simultaneously (Carlson et al., 2002; Thompson & Carlson, 2017). It is essential for understanding functions as models of dynamic events (Carlson et al., 2022) and for constructing and interpreting graphs of those functions. Although students can grasp covariation concepts as early as elementary and middle grades (Confrey & Smith, 1994; Johnson, 2015; Paoletti, Gantt, & Vishnubhotla, 2022; Thompson, 1994), engaging in covariational reasoning to conceive of situations in which quantities covary is not trivial without repeated practice and development (Carlson et al., 2002; Paoletti & Vishnubhotla, 2022).
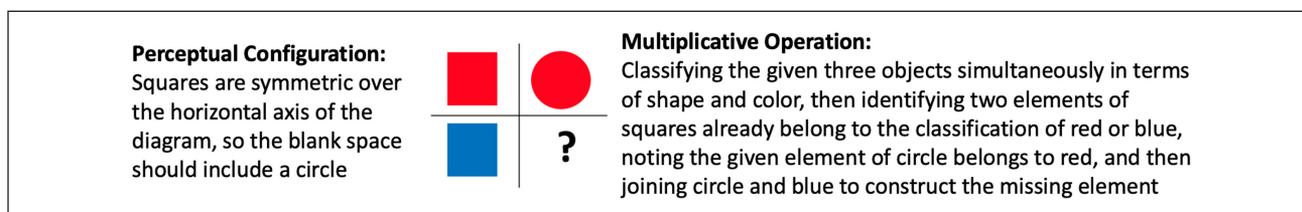
Researchers have identified various ways in which students reason covariationally (e.g., Carlson et al., 2002; Ellis et al., 2020; Thompson & Carlson, 2017). Our focus is not on distinguishing nuanced differences in covariational reasoning but on general categories of meanings, some of which include any form of covariational reasoning. For clarity and example production, we draw on empirical examples involving a particular form of covariation: gross coordination of values (Thompson & Carlson, 2017). *Gross coordination* involves attending to overall changes in two quantities' values (i.e., increase or decrease) relative to each other without considering specific values. For example, a person could describe the relationship between the height and volume of water in a filling bottle by noting that height increases as volume increases.

Multiplicative Object

The concept of uniting two distinct attributes into a single entity originates from Inhelder and Piaget's (1964) work on how children classify objects with two attributes, such as shape and color. They identified two ways of thinking: one is based on perceptual features, treating a matrix arrangement as an incomplete pattern, and the other on using a multiplicative operation to coordinate two classes logically (see examples in Figure 1). This cognitive process involves integrating two distinct attributes into a single conceptual object—what later work referred to as a *multiplicative object*.

Figure 1

A Matrix Diagram Derived From the Narratives of Inhelder and Piaget (1964) and Examples of Ways of Thinking When Identifying the Missing Element

Perceptual Configuration:
Squares are symmetric over the horizontal axis of the diagram, so the blank space should include a circle

Multiplicative Operation:
Classifying the given three objects simultaneously in terms of shape and color, then identifying two elements of squares already belong to the classification of red or blue, noting the given element of circle belongs to red, and then joining circle and blue to construct the missing element

Saldanha and Thompson (1998) built on this foundational idea and identified that covariational reasoning requires cognitively uniting two attributes into a multiplicative object that has both attributes simultaneously (see also Thompson, 2011). For example, an individual could conceive of a coin as having both a specific temperature and density at the same time. This requires holding both attributes in mind and understanding their simultaneous variation. In dynamic contexts, such as heating the coin to its melting point, having constructed a multiplicative object enables the tracking of temperature variations while maintaining a persistent awareness of simultaneous density changes.

In the context of covariation, Thompson (2011) represented the multiplicative object formed by uniting two quantities' variations as $(x_e, y_e) = (x(t_e), y(t_e))$, where $x_e = x(t_e)$ represents a variation in the values of x and t_e represents variation in t through conceptual time over the interval $[t, t+e]$. For students to reason covariationally, they must unite x_e and y_e to construct the multiplicative object (x_e, y_e), which is constituted by both quantities persistently and simultaneously. As we illustrate in the following section, this conceptual object is the basis for graphical representations and points in coordinate planes.

Graphs and Coordinate Systems

We focus on graphs constructed within coordinate-system constraints that resemble those used in mathematical communities. However, students' meanings of points and associated coordinate systems develop over time through novel and creative acts. Their constructions may result in organizations unfamiliar to researchers and not easily classified within existing coordinate-system types (e.g., diSessa, 1991; Lee, 2017). Our findings provide insights into the novel coordinate systems that students construct when their meanings for points lead to unique ways of structuring the plane, even when tasks are situated within common coordinate-system constraints.

We thus distinguish between conventional and nonconventional Cartesian planes. *Conventional planes* adhere to established practices of horizontally and vertically oriented axes representing quantities, ensuring consistent and efficient mathematical communication through social negotiation (Moore, Silverman, et al., 2019). *Nonconventional planes* deviate from these practices but can be equally valid structurally. Students' use of nonconventional planes is to be expected, must be recognized, and should be honored because conventional practices are learned through repeated experiences constructing a concept and communicating that concept among a group of individuals.

Lee et al. (2020) distinguished between spatial and quantitative coordinate systems. *Spatial coordinate systems* describe object locations within a real-life situation where a coordinate system is overlaid (Figure 2a). In contrast, *quantitative coordinate systems* involve conceiving of, disembedding, and projecting quantities onto a new space (Figure 2b). The key distinction lies in their goal-oriented activities. Spatial coordinate systems organize the figurative material of a present situation (e.g., a bike's path or physical locations on a map), whereas quantitative coordinate systems focus on acts of re-presentation, often in the absence of figurative material (Lee et al., 2020; Liang & Moore, 2021).

To characterize students' graphing actions and meanings for points, we distinguish between representation and re-presentation. Following von Glasersfeld (1995) and extensions of his perspective (Hackenberg, 2010b; Izsák et al., 2008; Liang & Moore, 2021), we use "representation" to refer to a mode of display (e.g., a graphical representation) or a figurative

Figure 2

(a) Illustration of Lee et al.'s (2020) Spatial Coordinate System (b) Quantitative Coordinate System
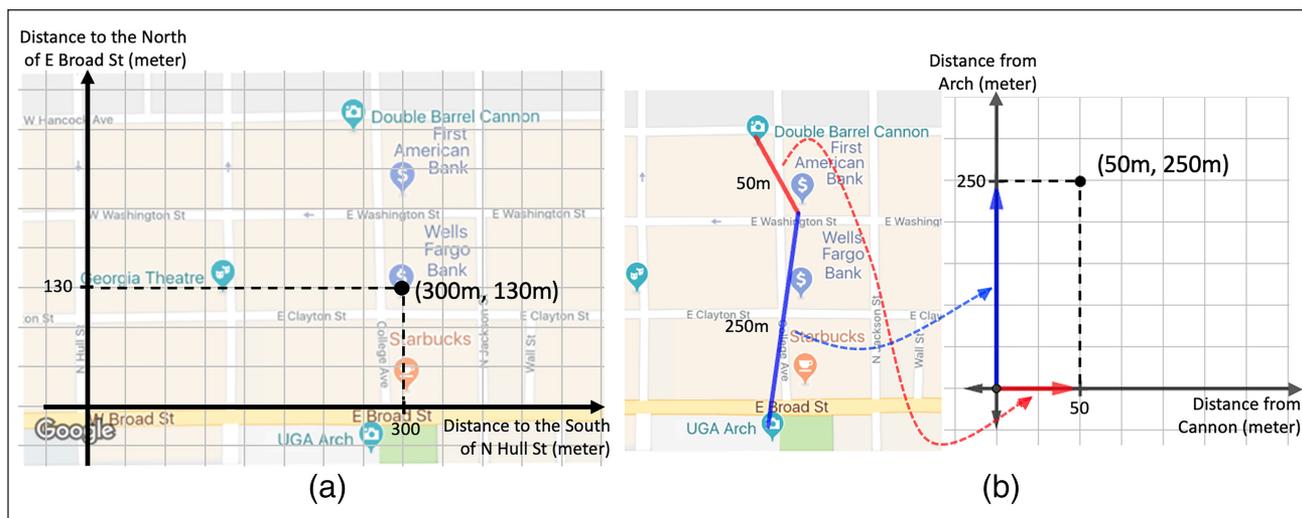
object within that display (e.g., a point on a Cartesian plane). In contrast, we use "re-presentation" to refer to the act of taking an object or attribute from one context and presenting it in another, often to preserve its structural equivalence as determined by the person doing the re-presenting.

For example, Lee et al.'s (2020) quantitative coordinate system differs from a spatial coordinate system because it involves an act of re-presenting quantitative properties—such as the length of a line segment or the volume of water—by graphically depicting them in a new system, beyond simply constructing referent axes. Underscoring our study's purpose and contribution, students' acts of re-presentation are a critical area of study for mathematics educators seeking to understand how students construct meanings for objects they come to understand as constituting representational systems. Researchers generate useful constructs and categories to characterize different mathematical objects, and Lee et al.'s (2020) quantitative and spatial coordinate systems are apropos examples of these. However, students' reasoning is often not as clean and defined as these constructs.

Related to acts of re-presentation, Moore and Thompson (2015) introduced *emergent shape thinking* to describe how a person envisions a graph "*simultaneously* as what is made (a trace) and how it is made (covariation)" (p. 785). This involves re-presenting two interdependent quantities' magnitudes on each axis, forming a multiplicative object, and generating a graph through the simultaneous coordination of covarying quantities (see Paoletti et al., 2023, for a local instruction theory on emergent shape thinking). Researchers have shown this to be a productive way of thinking about graphs when attempting to re-present covariational relationships (Frank, 2017; Moore, Stevens, et al., 2019; Paoletti et al., 2023). However, envisioning graphs as composed of points that record two covarying quantities is not a trivial meaning to construct (Frank, 2016, 2017; Saldanha & Thompson, 1998; Thompson et al., 2017). Reasoning emergently requires not only constructing a multiplicative object but also coordinating the two attributes of that object along with its representational object (e.g., a point) to re-present particular mathematical attributes (e.g., one quantity increasing as the other quantity decreases). The emphasis in school mathematics on plotting points as a procedure (e.g., over and up, Figure 3a) complicates this by leading to meanings that foreground taking physical actions rather than conceiving of a multiplicative object as simultaneously representing the two quantities' magnitudes located on axes (Moore, 2014; Figure 3b).

Hence, researchers and educators must examine students' cognitive processes, recognizing that their meanings of points may be confined to ordered pairs or spatial placements that do not symbolize or unite quantities' magnitudes. Constructing an object that represents two quantities as a single entity demands deeper comprehension beyond procedural knowledge. Students may proficiently plot points yet struggle with quantitative interpretations. This highlights the need to identify the specific multiplicative objects they are forming. A key challenge in mathematics education research is understanding how instruction can help students develop a meaning of points as multiplicative objects, a process closely tied to emergent shape thinking (Paoletti et al., 2023). Although this study does not directly address that challenge, it contributes by detailing the

Figure 3

(a) Plotting a Point as a Procedure (i.e., Over and Up) (b) Plotting a Point as a Projection of Two Quantities' Magnitudes
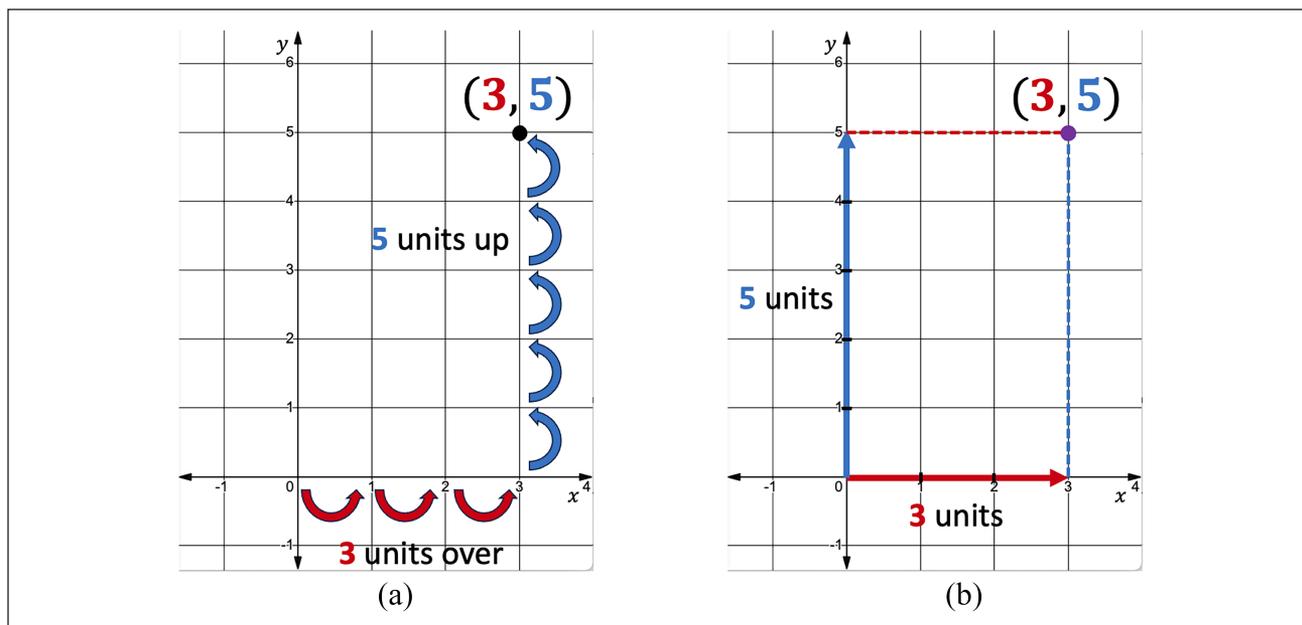
attributes and aspects students attempt to re-present when constructing graphs. The models of student thinking developed in our framework provide researchers and educators with tools to inform future research and instructional strategies that foster productive graphing meanings.

Methods

The data for this study were collected from two teaching experiments (Steffe & Thompson, 2000) at a public middle school in the southeastern United States. Participants included two seventh-grade students (Ella and Zane, age 12) and two sixth-grade students (Mike and Naya, age 11), all of whom volunteered with parental permission. The seventh graders were African American and the sixth graders were White. We use gender-preserving pseudonyms for all students.

Data Collection: Teaching Experiments

We explored middle school students' graphical meanings as they engaged in tasks emphasizing quantitative and covariational reasoning in experiential contexts. To develop tools useful for characterizing and supporting students, we conducted teaching experiments to construct second-order models[1] of their mathematical understanding and observe how their thinking evolved (Steffe & Thompson, 2000). The learning objectives focused on leveraging quantitative and covariational reasoning to support students' development of productive meanings for graphs, including emergent shape thinking. We have documented the participants' developmental shifts elsewhere (Tasova, 2022; Tasova & Moore, 2021); this article examines the different meanings of points observed during specific moments of the teaching experiments.

In spring 2019, Ella and Zane participated in six paired teaching sessions, after which Zane completed 10 individual sessions when Ella withdrew for family reasons. Each session lasted about an hour. In summer 2019, Mike and Naya participated in 14 paired sessions, each lasting approximately 50 min. Within paired sessions, students occasionally completed tasks individually. Balancing paired and individual sessions allowed us to iteratively develop and test models of student thinking while leveraging the benefits of each setting (Steffe & Thompson, 2000; Steffe & Ulrich, 2020; Ulrich et al., 2014). Group settings can stimulate thinking but may obscure individual contributions, whereas individual sessions allow the development of finer grained models of each student's thinking. We adjusted the setting as models evolved, ensuring rigor by generating data for hypothesis development and testing. In all sessions, the first author served as the teacher-researcher (TR), with other team members acting as witness-researchers.

Data Management and Analysis

We captured the teaching sessions on video to record participants' exact words, gestures, and drawings. Two video cameras were used: one for a wide-angle view of the students from the front and another for a focused view of their activities from above (Figure 4). We also collected screen recordings from a tablet device displaying animations. After each session, we transcribed the videos and digitized students' written work and the notes taken by research team members.

Ongoing and Retrospective Analysis

We conducted both ongoing and retrospective conceptual analyses (Steffe & Thompson, 2000). During ongoing analysis, we formulated and tested hypotheses about each student's thinking on the basis of their task engagement. To refine these hypotheses and build models, we engaged students in varied situations across tasks and settings (i.e., group and individual sessions). We adjusted subsequent sessions using our ongoing analysis to systematically investigate our interpretations. Throughout this process, we took detailed notes on student engagement, interpretations of their actions, and decisions for subsequent session design, providing valuable resources for retrospective analysis.

In the retrospective analysis, we revisited the data after the teaching experiments to build and revise the ongoing models developed during student interactions. Using an open and axial coding approach (Strauss & Corbin, 1998) combined with conceptual analyses (Thompson, 2008; von Glasersfeld, 1995), we implemented an iterative process to generate descriptors of student meanings and then conduct coding analysis of the video recordings to provide reliable explanations of students' activities (i.e., second-order models). This process involved multiple passes through the recordings, with each pass informing subsequent passes and each second-order model informing the other second-order models.

First, we analyzed each student's data, including ongoing analysis results, to answer: "What mental operations must be carried out to see the presented situation in the particular way one is seeing it[?]" (von Glasersfeld, 1995, p. 78). We identified characteristics of students' quantitative and nonquantitative interpretations, meanings for points, and graphical

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[1] A *second-order model* refers to a hypothetical model of a student's mathematics that a researcher constructs through intentional interactions with that student. Steffe and Thompson (2000) distinguished second-order models from a researcher's first-order mathematics to clarify that both are products of the researcher's thinking: First-order models reflect the researcher's own mathematics, whereas second-order models represent the mathematics the researcher attributes to someone else.

Figure 4

Image of the Room Setup
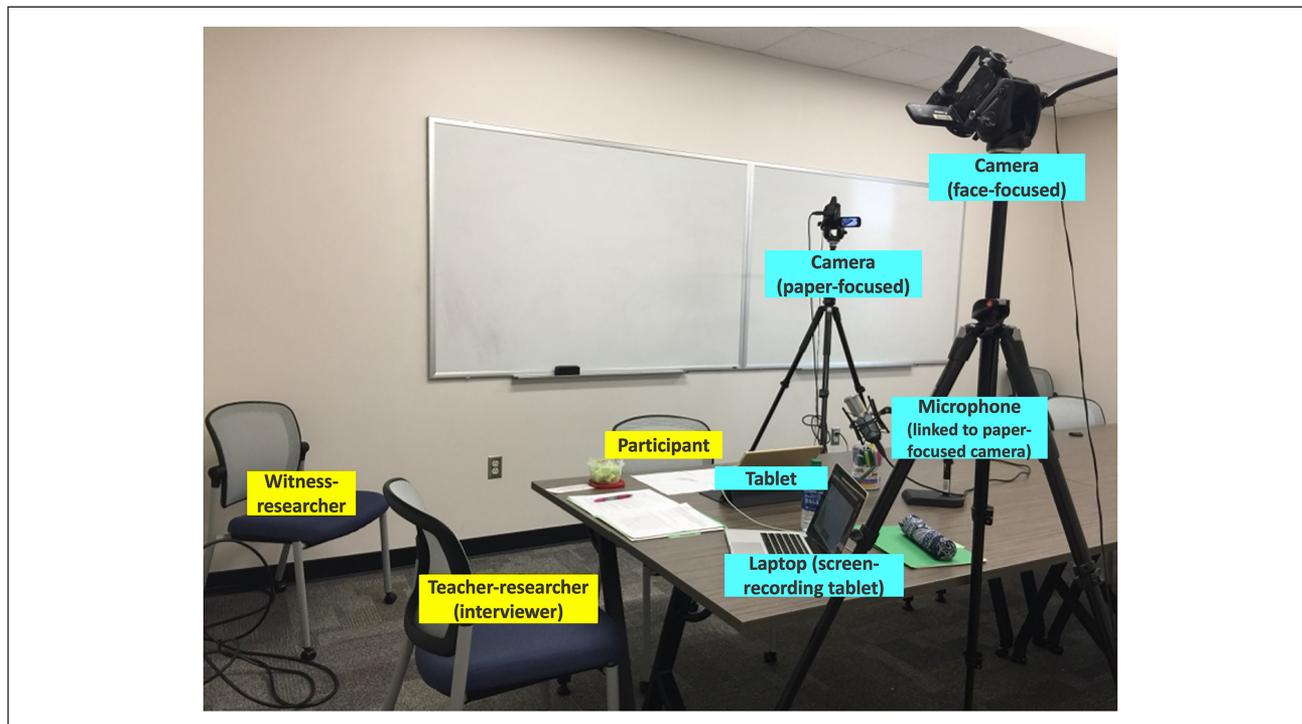
actions. By comparing these characteristics across participants, we generated categories of students' meanings, including those for points on the coordinate plane. To distinguish between quantitative and nonquantitative interpretations, we followed established guidelines (Moore et al., 2022, 2024; Thompson, 2011) to gain insights into the extent to which a student is reasoning about measurable attributes. We paid particular attention to the extent to which a student's actions indicated that they held a magnitude in mind, constructed a unit for measuring that magnitude, established a stable frame of reference for it, and formed comparisons between magnitude states.

With student meaning categories established, we conducted another pass through the data to code each session accordingly. This process allowed us to explore the effectiveness of our categories in characterizing student actions. During this phase, we identified instances that were not adequately captured by the categories of student meanings, leading us to refine and modify them so that they more viably captured the students' actions. In yet another round of retrospective analysis, we examined connections among categories and identified patterns and relationships. We took extensive notes and created memos to assist us when new analyses challenged previous conclusions. At times, we revisited and refined our second-order models to better explain emerging connections and implications. For example, in attempting to connect or explore the implications of various categories, we identified the need for more detailed articulations of our second-order models. Both authors engaged in collaborative discussions throughout the process to reconcile differences and reach consensus. See Figure 5 for an overview of the analysis process.

Our iterative, multiphase analyses resulted in the framework we present here, with its components emerging and stabilizing at different stages of retrospective analysis. This process aligns with Steffe and Thompson's (2000) perspective on viability and generalizability in teaching experiments and the development of second-order models of students' mathematics. Only through developing multiple second-order models can a researcher produce, elucidate, and refine characteristics that are uniquely viable to one of them; a second-order model of a participant's thinking becomes a tool for developing a second-order model for a different participant, and the process of developing subsequent second-order models better positions the researcher to clarify additional aspects of previously generated second-order models. Similarly, a researcher can identify and refine shared characteristics among second-order models as they encounter student data compatible with previously generated second-order models. The categories and characteristics that emerge and stabilize serve as "useful in organizing and guiding our experience of students doing mathematics" (Steffe & Thompson, 2000, p. 304). A framework like the one given here can then form the foundation for future studies to explore their broader applicability among the diverse population of learners in this world. We return to this point in the closing of the article.

Figure 5

Ongoing and Retrospective Analyses Overview
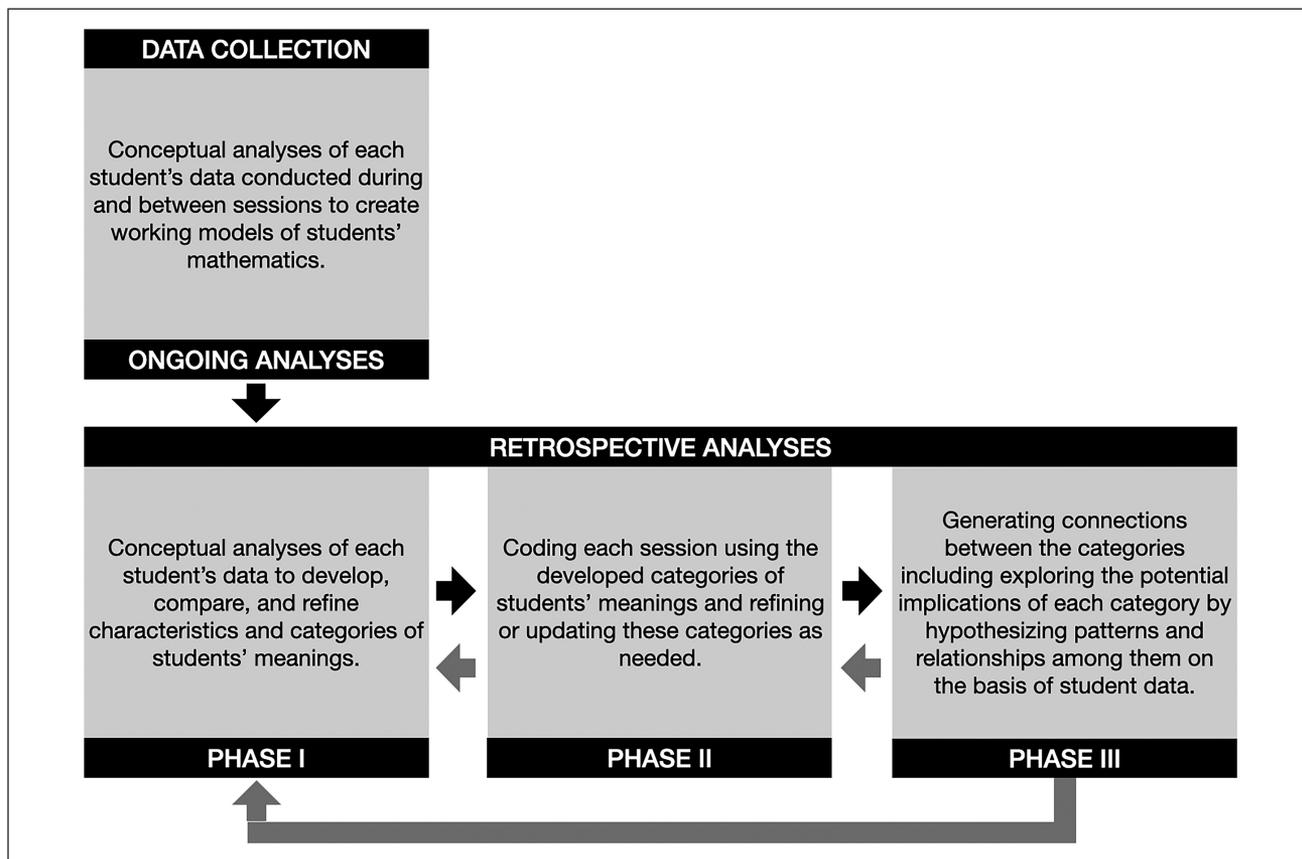
Reflexivity in Teaching Experiment Methodology

Reflexivity—"the need to be explicit about biases and assumptions and also about how the research has affected and even changed the researcher" (de Freitas et al., 2017, p. 169)—played a critical role in our methodology. Throughout the teaching experiment, the TR maintained a balance between respecting participants' personal realities, actions, and contributions and attempting to gain insights into and potentially influence those realities. This approach sometimes required the researchers to engage in intentional actions to influence participant actions, particularly when testing hypothesized models of student thinking by exploring the limits of those models. Even in such cases, the researchers maintained a position of empathy, informed curiosity, and inquisitiveness (Hackenberg et al., 2024). The TR continuously explored the viability of our models of student thinking, recognizing that the TR's actions influenced participants' thinking. Our goal in this article is thus to report on the models of student thinking that this approach enabled us to develop, rather than provide an explanatory account of how TR actions influenced participant thinking.

Radical constructivism, which underpins our methodology, emphasizes self-reflexivity (Hackenberg et al., 2024). Hackenberg et al. (2024) emphasized that TRs should engage in productive doubt of their models, subjecting them to scrutiny and viewing them as tentative interpretations that could be revised with further analysis. This requires that researchers reject claims of objectivity or correctness in their models, always considering alternative models and scrutinizing their models through ongoing and retrospective analyses. The models are never to be interpreted as claims about what is occurring in the mind of the student. The models presented here are hypotheses of student thinking, built on existing viable models and our own observations. They warrant further scrutiny and revision—a point to which we return in the closing of this article.

Task Design and Implementation

In each session, as students engaged in graphing activities, the TR prompted them to explicitly discuss their thinking. The TR and research team designed tasks that were based on ongoing models of student thinking, occasionally drawing on existing research on graphing covarying quantities (e.g., Frank, 2017; Moore & Thompson, 2015; Stevens et al., 2017;

Thompson & Carlson, 2017). We created the tasks using dynamic geometry software (GeoGebra) and displayed them on a tablet.

Each task in our teaching experiments involved an experiential context (e.g., riding a bike) that students had to make sense of before engaging in graphing. We first examined how students conceived of quantities within each situation, then prompted them to proceed to graphing only after establishing a stable model of their understanding. In general, students exhibited a mix of quantitative and nonquantitative understandings of the situations. Consistent with prior research (Izsák, 2000, 2003; Czocher et al., 2022; Moore & Carlson, 2012), we found that students' graphing activities were re-presentations of these understandings. Their acts of re-presentation provided further data to refine our models of how they conceptualized situations. Importantly, we observed no evidence in their re-presentational activities that required a fundamental revision of our model of their understanding of the situations.

In the remainder of this section, we illustrate the tasks used in our results to showcase student reasoning. Digital versions of all main tasks used in the teaching experiment are available online (https://www.geogebra.org/m/fww2c5mz).

Downtown Athens Task

The Downtown Athens Task was an adapted version of Sketching Graphs from Pictures: Particles and Paths (Swan, 1985). It included a map of downtown Athens, GA with seven labeled locations: UGA Arch (Arch), Double-Barreled Cannon (Cannon), First American Bank, Georgia Theater, Wells Fargo Bank, Statue of Athena, and Starbucks (see https://www .geogebra.org/classic/mnfdmtzg). The task also included a Cartesian coordinate plane with axes labeled "Distance from Cannon" and "Distance from Arch." Seven unlabeled points on the plane represented each location's corresponding distances from these landmarks. We asked participants what each point might represent to observe their meanings and explore their meanings of points in what we perceive to be a Cartesian coordinate plane. Although the task was designed using Euclidean distances, we ensured that discussions explored students' interpretations of "distance," acknowledging its multiple possible meanings. Given the street layout of the map, the context could naturally lend itself to a taxicab or path-based interpretation of distance. For this reason, we took students' interpretations seriously and negotiated with them what was meant by "distance" in the task, ensuring alignment with the intended interpretation while honoring their reasoning.

Downtown Athens Bike Task

The Downtown Athens Bike Task used the same map of downtown Athens but highlighted a straight street (Clayton St. or College Ave.) with two nearby landmarks: the Arch and Cannon (see https://youtu.be/bTRJskGP03U). An animation displayed a bike moving back and forth along this street. This task was inspired by similar activities in Swan (1985; Going to School), Ponte (1984; Journey to School), Stevens et al. (2017; Going Around Gainesville), and Saldanha and Thompson (1998; City Travels). Our goal was to explore how students conceived of the relationship between two covarying quantities. Students were asked to sketch the relationship between the bike's distance from Arch and Cannon as it moved. Because the distances from Arch and Cannon vary together—increasing or decreasing simultaneously as the bike moves—students would probably produce an upward-sloping line graph. Because the graph's shape closely resembles the bike's path in this scenario, we designed a second version in which the bike traveled along College Ave. from north to south, creating a graph that no longer matched the bike's physical path on the map.

Although the Downtown Athens Task included a static situation whereas the Downtown Athens Bike Task was dynamic, both tasks involved considering quantities that were distances between locations. This allowed students to visualize these quantities as segments on the graph. Students could use these segments as re-presentations of quantities' magnitudes and operate on them (e.g., disembedded, translate, partition, etc.).

Swimming Pool Task

The Swimming Pool Task, adapted from Swan (1985), included a dynamic diagram of a pool (see Figure 6a or https:// youtu.be/SWIOqzYFzT0); students could fill or drain the pool by dragging a point on a slider. We designed the task to support students in reasoning with the relationship between two continuously covarying quantities: amount of water and depth of water in the pool. We asked students to describe how amount of water varied in relation to depth of water as we filled the pool. We gave them grid paper and marks on the right side of the pool (Figure 6b) to offer some support in equally partitioning depth of water as well as amount of water. We then asked students to sketch a graph showing the relationship between amount and depth of water.

Results

We illustrate students' distinct graphing meanings according to our analysis of their actions during the teaching experiment. The students' meanings are summarized in Table 1. At the most general level, we categorized these meanings as either iconic or quantitative. Iconic meanings involve iconic translation and transformed iconic translations. Coordinate

Figure 6

(a) A Diagram of the Pool and (b) A Diagram of the Pool With Grid

(a)   (b)
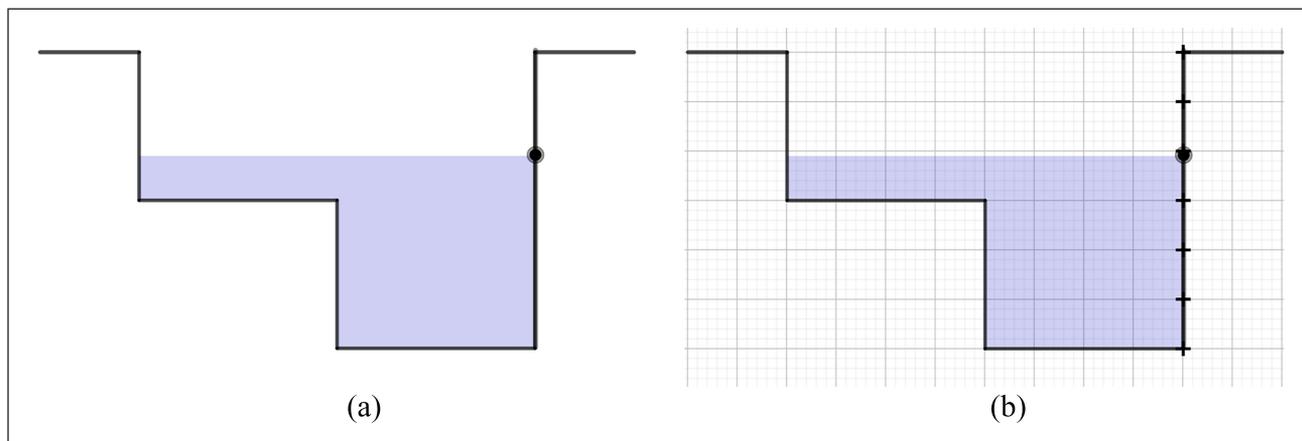
system structures are not intrinsic to iconic meanings, and thus the two coordinate systems introduced by Lee et al. (2020) are not relevant to these meanings. Instead, these meanings align with Moore and Thompson's (2015) notion of static shape thinking, in which a graph is viewed as an independent object with indexical associations to the situation.

Quantitative meanings involved a point representing (a) a single quantity, (b) a spatial-quantitative multiplicative object, and (c) a quantitative multiplicative object. Some students re-presented a single quantity within a quantitative coordinate system (Lee et al., 2020; see Table 2). Their plotted points and graphs aligned with emergent graphical shape thinking, though in a variational form. Other students engaging with spatial-quantitative multiplicative objects demonstrated emergent graphical shape thinking. This occurred within a coordinate system integrating quantitative and spatial properties (Lee et al., 2020). Students also reasoned about a graph as an emergent trace of covarying quantities with the meaning of a quantitative multiplicative object, but they did so in the context of a quantitative coordinate system.

Next, we present empirical data for each meaning. We briefly describe iconic translation, a well-established concept in mathematics and science education (e.g., Clement, 1989; Monk, 1992; Johnson et al., 2020). We then focus on transformed iconic translations and the various quantitative meanings because of their contributions as novel models of student thinking.

Iconic Meanings

Students' iconic meanings involved envisioning points on a plane (Figure 7b) as a location or object that is physically or figuratively associated with a location or object in the situation (e.g., map of Downtown Athens, Figure 7a). With iconic

Table 1

Overview of the Results

Category and subcategory	Description	Example
Iconic		
Iconic translation	Making a perceptual association between objects or locations in the situation and the points on the plane	Student drawing a map with locations or objects on the plane as it appears in the situation
Transformed iconic translation (rigid and nonrigid)	Making a perceptual association between a transformed version of the situation and the points on the plane	Student adjusting shapes of a map with locations or objects to fit the plane
Quantitative		
Nonunited points	Conceiving of a point on the plane through re-presenting a single quantity's magnitude at a particular time or state	Student plotting a single point on the plane representing the bike's distance from a location at a particular time or state
Spatial-quantitative multiplicative object	Conceiving of a point on the plane as an object or location through re-presenting the object or location and specified quantitative properties	Student plotting a point representing the physical bike on the plane by coordinating its distance from Arch and Cannon
Quantitative multiplicative object	Conceiving of a single point on the plane through re-presenting two quantities' magnitudes simultaneously	Student plotting a point on the plane representing the bike's distance from Arch and Cannon

Table 2

Connections to and Integration of Existing Quantitative and Covariational Reasoning Constructs

Subcategory	Coordinate systems	Connection to covariational reasoning
Nonunited points	Quantitative coordinate system	Considering how one quantity changes over time (variational form of emergent shape thinking)
Spatial-quantitative multiplicative object	Blend of spatial and quantitative coordinate systems	Considering how quantities change together (emergent shape thinking)
Quantitative multiplicative object	Quantitative coordinate system (conventional and nonconventional)	

meanings, what is being translated are objects or locations, not their measurable attributes (i.e., quantities). We distinguish between (a) iconic translation and (b) transformed iconic translation. We briefly illustrate iconic translation and elaborate on transformed iconic translation. Students drawing on this meaning did not re-present a quantitative multiplicative object, because the point on the plane does not unite the magnitudes of two quantities.

### Point as a Location or Object and Iconic Translation

Iconic translation occurs when students assimilate points on a plane as locations or objects. Researchers introduced this notion to characterize when individuals interpret graphs as literal pictures of a situation (e.g., Clement, 1989; Monk, 1992). In our teaching experiments, some students associated the perceptual or spatial features of objects in the situation (Figure 7a) with corresponding points on the plane (Figure 7b). For example, Mike and Naya employed this meaning by assimilating the point on the horizontal axis in Figure 7a as the physical Arch because both the map (Figure 7a) and the plane (Figure 7b) positioned it in the southernmost location. Drawing on the same iconic translation meaning, Mike experienced a perturbation regarding the location of Starbucks relative to Arch, stating, "on the map [*pointing*], the Starbucks is right there [*pointing to the plane and circling a location indicated by the red arrow in Figure 7b*], not to the left." Similarly, he expected to see an object up and to the left of the Arch on the map because a corresponding point on the plane existed in that location, despite no object appearing on the map (see his circled location indicated by the blue arrow in Figure 7a). His perturbation illustrates how iconic translation involves coordinating the perceptual or spatial features so that points on the plane mirror their associated objects in the situation.

Point as a Location or Object and Transformed Iconic Translation

A strict iconic translation (such as the one by Mike and Naya) involved students translating from the situation to the plane (or from the plane to the situation) the perceptual features exactly as they were. A *transformed iconic translation*
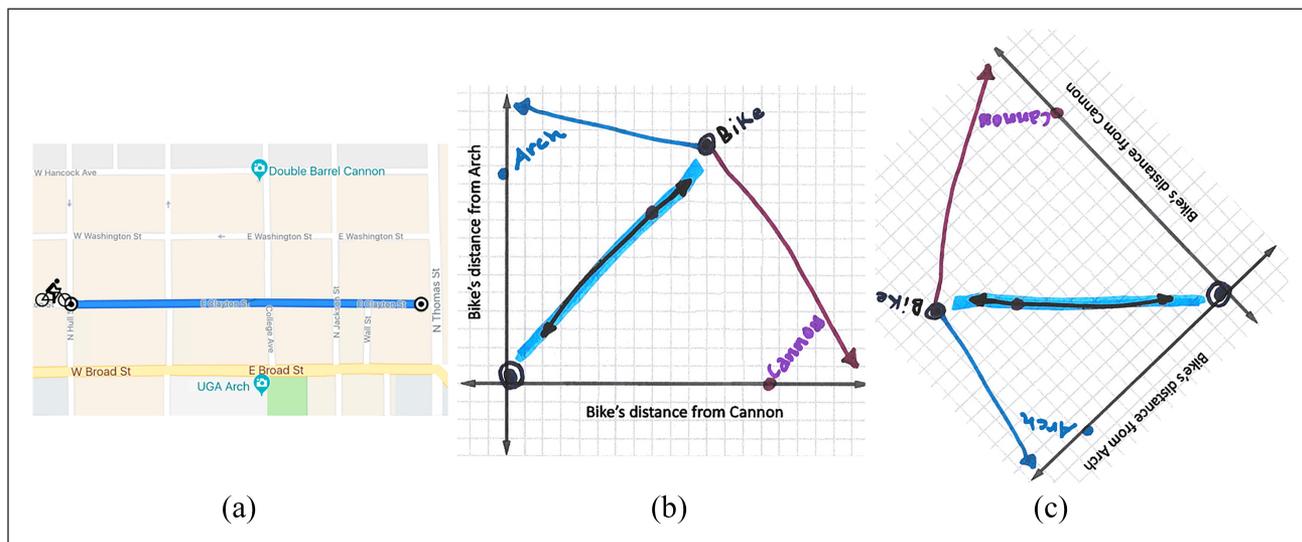
Figure 7

Mike and Naya's Activity in the Downtown Athens Task
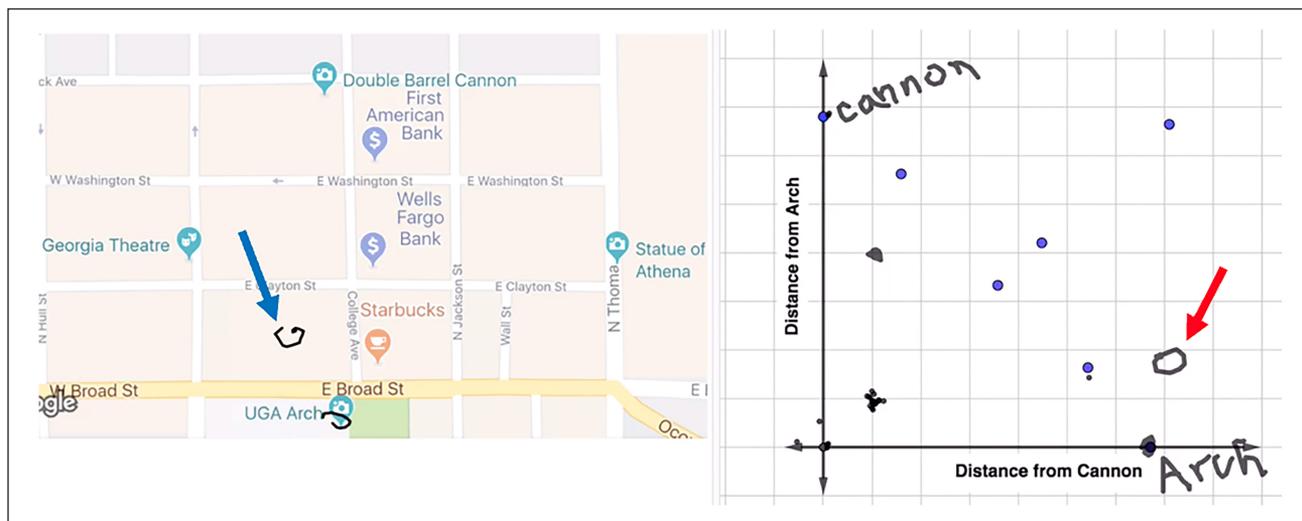
Note. Red and blue arrows were added for the reader's benefit.

Figure 8

(a) The Location of the Bike, (b) Zane's Second Draft in Downtown Athens Bike Task, and (c) Zane's Rotated Graph

(a)	(b)	(c)

involved students translating a transformed version of the perceptual features of the situation to the plane (or in the opposite direction). Students engaged mainly in two forms of transformation: (a) rigid and (b) nonrigid (deformable) transformations. *Rigid transformations* included rotating the page containing the picture of the situation (e.g., a map or a swimming pool) or the graph, flipping it vertically, or combining both actions. *Nonrigid transformations* involved altering an object's shape or configuration to preserve relative positions or angles when transferring it from the situation to the plane (or in the opposite direction). Nonrigid transformations may include stretching, compressing, bending, and other forms of deformation.

An Illustration From Zane's Teaching Experiment. Zane's activity in the Downtown Athens Bike Task exemplified a rigidly transformed iconic translation[2]. The TR asked him to sketch a graph showing the relationship between the bike's distance from Arch and Cannon as it traveled at a constant speed on the map (Figure 8a). Zane conceived of Arch and Cannon as locations on the vertical and horizontal axes, respectively, as implied by the labels. He drew a light blue segment moving upward from left to right, marking two dots—one at the origin and another labeled "Bike" (Figure 8b). These dots corresponded to the bike's path endpoints on the map, and the segment color matched the path. To explain his graph, Zane rotated the paper, aligning the blue segment horizontally with the "Bike" dot on the left (Figure 8c). This rotation made his graph identical to the map, assimilating his graph on the plane as the map itself.

An Illustration From Naya's Teaching Experiment. Naya's activity in the Downtown Athens Task exemplified a nonrigidly transformed iconic translation. She associated points on the plane's vertical and horizontal axes with the physical locations of Arch and Cannon, respectively (Figure 9b). She then made a perceptual association between the axes of the plane and the top and bottom of the map, imagining the horizontal axis of the plane at the top of the map and the vertical axis at the bottom (Figure 9a). Naya explained, "I think this [bottom of the map] is more like the *y*-axis, and this [top of the map] is like the *x*-axis. So, you put them together. Then, you have all this stuff in the middle," pointing to the overall locations on the map and to the points on the plane.

Naya conceived of the axes as boundary lines with objects "in the middle." She also imagined flipping the map vertically, placing Arch at the top and Cannon at the bottom (Figure 9c), stating, "once we flipped it [the map] around, all that stuff [top part] is down here [bottom]." This flip aligned points on the plane with their map locations, making her nonrigid iconic translation coherent.

She continued making perceptual associations between other points and locations. After flipping the map (Figure 9c), she noted that Arch, Starbucks, and Wells Fargo Bank formed a straight line from top to bottom in both the map and the plane (Figure 9b). Similarly, the Georgia Theater, Wells Fargo Bank, and the Statue of Athens were colinear in both (see yellow highlights in Figure 9c). Naya's meaning of the points included a nonrigidly transformed iconic translation, using perceptual features of the transformed map and plane to make sense of the points on the plane.

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[2] In this context, "translation" refers to how students perceptually and conceptually re-present features from one representation (e.g., a map or situation) onto another (e.g., a coordinate plane). This aligns with our earlier use of iconic translation, in which students assimilate perceptual features between representations. It is distinct from the geometric transformation known as translation, which refers to shifting a figure without rotation or distortion.

Figure 9

(a–b) Naya's Graphing Activity in Downtown Athens Task (c) Flipped Version of the Map Created for the Reader

(a)

(b)

(c)
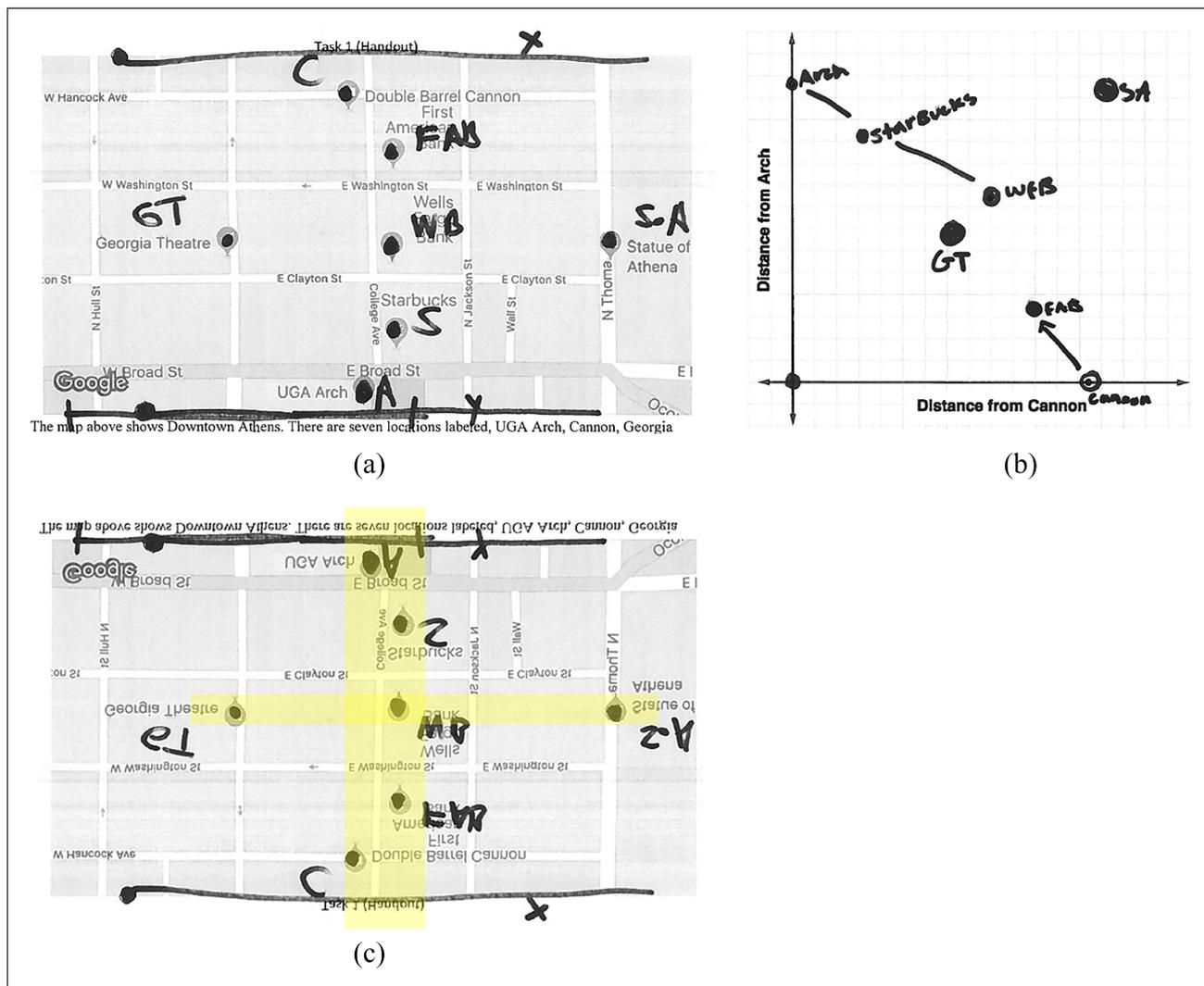
Note. Yellow highlights in Panel c were added for the reader's benefit.

Quantitative Meanings

In this section, we present and analyze students' quantitative meanings of points, categorized into three types: (a) a point representing a single quantity's magnitude (in relation to time; nonunited points), (b) a point representing a spatial-quantitative multiplicative object, and (c) a point representing a quantitative multiplicative object in conventional and nonconventional Cartesian planes. The first category includes a point representing one and only one quantity's magnitude (in relation to time), but the latter two involve a point representing a multiplicative objects consisting of two quantities' magnitudes and their relationships. We illustrate each category with empirical data from our teaching experiments.

Point as a Representation of a Single Quantity's Magnitude (Nonunited Points)

Nonunited points are instances when students interpret points on a plane in terms of re-presenting the magnitude of a single quantity at a specific event state. For example, in the Downtown Athens Bike Task, a student implementing a nonunited points interpretation would plot one point for the bike's distance from Arch at a specific state and another point for its distance from Cannon at that state. Although these points represent quantities, they do not function as multiplicative objects with respect to the quantities defining the Cartesian axes; each point pertains to a single quantity at a particular state. However, this does not mean the student has not conceptualized the point as a multiplicative object. Variational and covariational reasoning always relate to time, whether implicitly or explicitly (Paoletti & Moore, 2017; Patterson & McGraw, 2018; Thompson & Carlson, 2017). Moore (2025) further clarified that whether a point is a multiplicative object depends

Figure 10

(a) Mike's Graphs in Swimming Pool Task and (b) Swimming Pool Situation
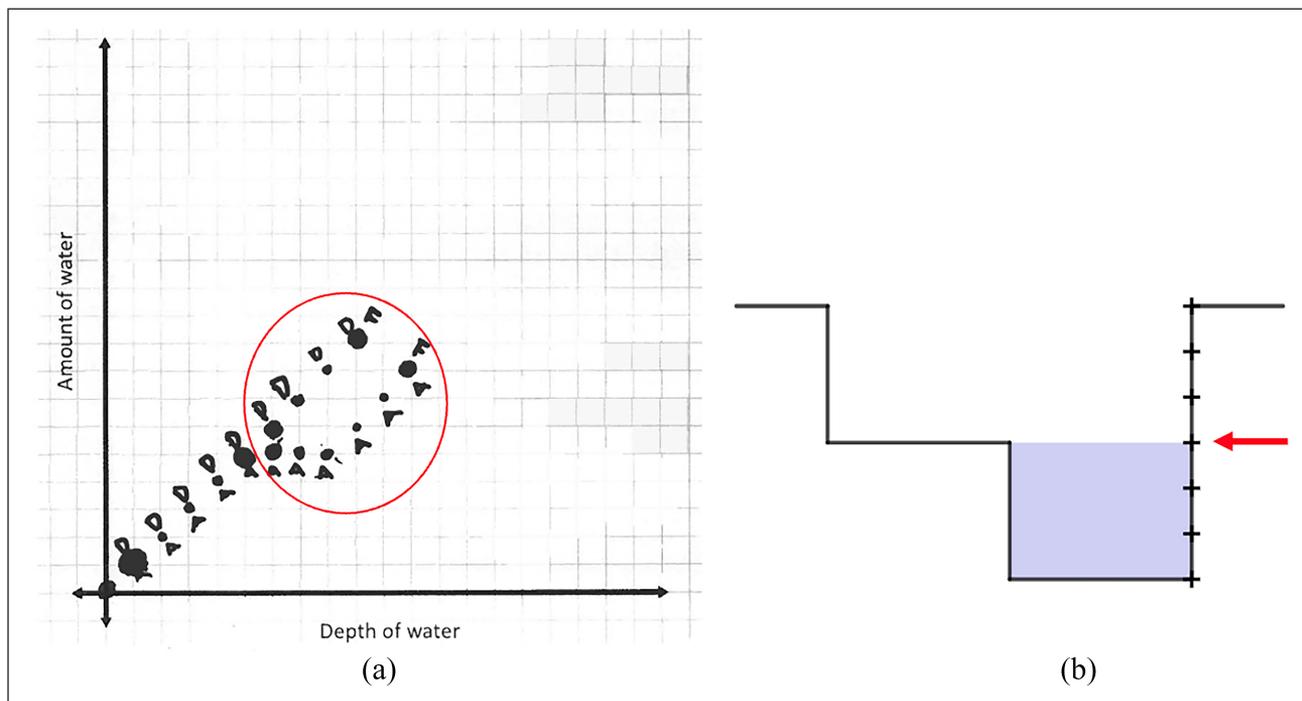
Note. Red circle and arrow were added for the reader's benefit.

on whether the student is reasoning about the quantity's variation with respect to experiential (passing) or conceptual (measured) time, with the latter being required to construct a multiplicative object. Because our focus is on the quantities defining the axes, we do not articulate nuances of the students' reasoning about time but instead use "states" to describe different statuses of a perceived experience in either experiential or conceptual time. We underscore that claiming a point is not a multiplicative object with respect to the coordinate axes does not imply the point is not a multiplicative object involving one of the quantities in association with event states conceived of in conceptual time.

Mike's activity in the Swimming Pool Task exemplifies nonunited points. He started his graphing by plotting points on the plane from the origin (Figure 10a). For the bottom half of the pool, he plotted points showing the relationship between the amount and depth of water (dots outside the red circle in Figure 10a, labeled A and D). For the top half of the pool, he plotted a point for each quantity for several states of the pool situation (dots within the red circle in Figure 10a).

Using the tick marks on the pool's right edge, Mike noted that depth "increases at the same amount each time and never changes," leading to his consistent plotting of depth points (Figure 10a). For the amount of water, he observed that after the third tick mark (red arrow in Figure 10b), "the amount of water gets twice as large as it was before." That is, he determined that the amount in the top half of the pool "changes twice as much as" the change in the amount of water for the bottom half for each tick mark. Thus, the amount of water points on the plane for the top half (dots in the red circle; Figure 10a) are plotted further from the vertical axis labeled "Amount of Water." He explained, "the amount of water [*tracing his finger on the dots labeled A in the red circle*] gets further away from the *y*-axis to show that there is more, and the depth of water stays the same, so it keeps going." In summary, each point Mike constructed represented a single quantity's magnitude at a particular state of the filling pool. This included his initial points, which he understood as two overlapping points, each indicating the re-presenting of a quantity's magnitude at a particular state.

Representing a Spatial-Quantitative Multiplicative Object

In this section, we illustrate a meaning for points in terms of re-presenting a spatial-quantitative multiplicative object, incorporating both spatial and quantitative features of a situation. Students' representation of a spatial-quantitative multiplicative object involved envisioning a point on the plane as a location or an object, with the location of the point determined by re-presenting the location or object's quantitative properties (e.g., gross comparison of two quantities' magnitudes). This meaning is similar to iconic translation in that students conceived of points as representing locations or objects within the given situation, rather than as entities signifying the disembedded magnitudes of both quantities (e.g., the bike's distance

Figure 11

(a) Ella's Segments in the Situation and (b) Ella's Segments in the Plane

(a)   (b)

from Arch and Cannon). However, unlike iconic translation, this meaning involved identifying the quantitative features of a location or object in the situation (i.e., its distance from Arch and Cannon), and ensuring that these quantitative properties were preserved when re-presenting the location or object on the plane. As we illustrate, students who re-presented a spatial-quantitative multiplicative object often drew segments in the situation to indicate quantities' magnitudes and made gross additive comparisons among those apparent magnitudes. Moreover, they transformed those magnitudes from the situation to the graph, as well as vice versa. But they also conceived of some elements in the plane as physical locations or objects that they intended their points to preserve.

An Illustration From Ella's Teaching Experiment. Ella's activity in the Downtown Athens Task exemplified a spatial-quantitative multiplicative object meaning. She formed, from our perspective, a two-center bipolar coordinate system by comparing two quantities' magnitudes. Ella conceived of Arch and Cannon as locations on the vertical and horizontal axes, respectively, as implied by the labels (orange dots at each axis end in Figure 11b). She then coordinated radial distances between points on the plane—perceived by her as places on the map—and their respective distances from Arch and Cannon. For example, Ella labeled a point "FAB" on the plane to indicate the point that represented the physical First American Bank (Figure 11b). She reasoned that the orange and blue line segments on the plane showed that First American Bank is closer to Cannon and farther from Arch, similar to their positions on the map (Figure 11a). Thus, she re-presented First American Bank's distance from Arch and Cannon by the respective distances from the axes. Ella anticipated preserving the quantitative features of the situation in the plane, indicating that her meaning of the points was as spatial-quantitative multiplicative objects.

An Illustration From Mike's Teaching Experiment. Mike's activity in the Downtown Athens Bike Task also exemplified the spatial-quantitative multiplicative object meaning. He initially drew a straight, upward-sloping line from left to right (Figure 12b) to re-present the bike's path "where the bike travels," and explained, "I drew this line and I imagined East Clayton Street." He added tick marks and dots along the line "to represent like where the bike could be." This suggested Mike conceived of each point on the graph as the physical bike moving on its path on the plane.

Unlike iconic translation (see Zane's graph in Figure 8), Mike determined the bike's position on the plane on the basis of its distances from Arch and Cannon. For example, when the bike reached a location labeled 2 on the map (Figure 12a), Mike remarked that "it [the bike] would go right here [*pointing to the tick mark labeled 2 in Figure 12b*]" because the bike's distances from both Arch and Cannon were minimums at that point. Similarly, he compared distances for other locations (labeled 1 and 3) to establish their corresponding positions on the graph.

Mike drew horizontal and vertical line segments to indicate his re-presenting the bike's distances from Arch and Cannon, respectively. He re-presented the bike's distance from Arch with the point's distance from the vertical axis and the bike's distance from Cannon with the point's distance from the horizontal axis. Notably, the horizontal segments were consistently shorter than the vertical ones, reflecting Mike's observation that the bike's distance from Arch was always less than its distance from Cannon.

Figure 12

(a) Downtown Athens Bike Task and (b) Mike's Graph
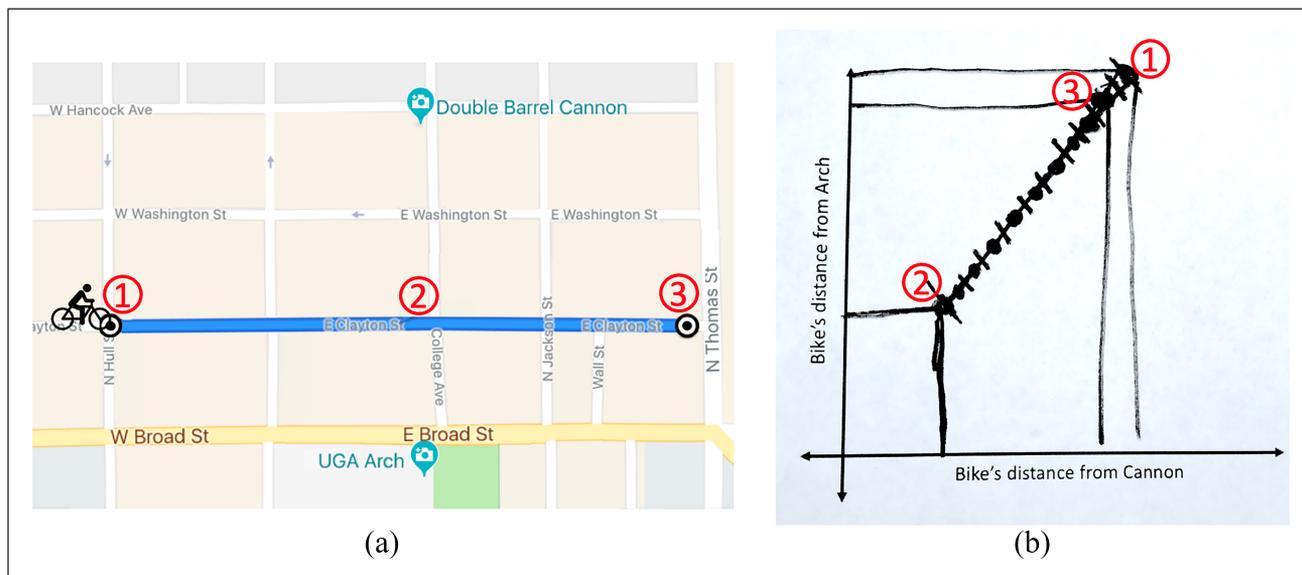
Note. Numbered labels are added for the reader's benefit.

Mike's activity suggested that his meaning of the points involved the bike's quantitative properties and the gross comparison of two quantities' magnitudes. Unlike Zane, whose meaning we illustrated earlier, Mike did not use a direct iconic translation of the path. Instead, he focused on preserving and re-presenting the bike's quantitative properties on the plane.

Representing a Quantitative Multiplicative Object

In this section, we illustrate a meaning for a point as representing a quantitative multiplicative object—a conceptual object that is formed by uniting in the mind two or more quantities' magnitudes simultaneously (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017). That is, students using a point as a representation of a quantitative multiplicative object envisioned a single entity on the plane as symbolizing the result of re-presenting two quantities' magnitudes simultaneously. Our study revealed that students interpreted the coordinate plane in two primary ways. In one interpretation, aligned with the conventional Cartesian plane, students conceived of the axes as number lines on which they re-presented the quantities' magnitudes and united in the plane (Figure 13a). They conceived of the origin as a reference point for each axis from which to measure the distance. Students created a point by taking two orthogonal magnitudes along the axes and creating projections. In turn, the space of the graph is inconsequential beyond providing a plane to create a point by joining the projections.

In the second interpretation, which we consider an example of a nonconventional coordinate system, students' meaning of a point was operationally equivalent to a quantitative multiplicative object, but they interpreted the axis labels differently than on a conventional Cartesian plane. Students did not conceive of the axes as number lines. Rather, they conceived of the vertical axis as a reference ray, and they re-presented the magnitude of the ray's labeled quantity using horizontal distances from this ray. Similarly, they conceived of the horizontal axis as a reference ray, and they re-presented the magnitude of the ray's labeled quantity using vertical distances from this ray. For example, horizontal distances from the vertical axis labeled "Amount of water" represented the magnitudes of amount of water (i.e., the blue segment in Figure 13b). Similarly, vertical distances from the horizontal axis labeled "Depth of water" represented the magnitudes of depth of water (i.e., the red segment in Figure 13b).

We underscore that in both conventional and nonconventional coordinate systems, points represented quantitative multiplicative objects, uniting the two quantities' magnitudes simultaneously. Both interpretations give identical meanings to points; their difference is in how points are conceived quantitatively with respect to the rays forming the Cartesian axes. To illustrate these two interpretations of quantitative multiplicative objects, we present examples from Zane's and Mike's work during the Swimming Pool Task.

An Illustration From Zane's Teaching Experiment (a Quantitative Multiplicative Object in a Cartesian Plane). Zane began his graphing activity in the Swimming Pool Task by drawing tick marks on each axis and plotting points corresponding to two related tick marks (see his color-coded points and tick marks in Figure 14a and Figure 14b). He then

Figure 13

(a) Quantitative Multiplicative Object in a Cartesian Plane and (b) Quantitative Multiplicative Object in a Nonconventional Cartesian Plane
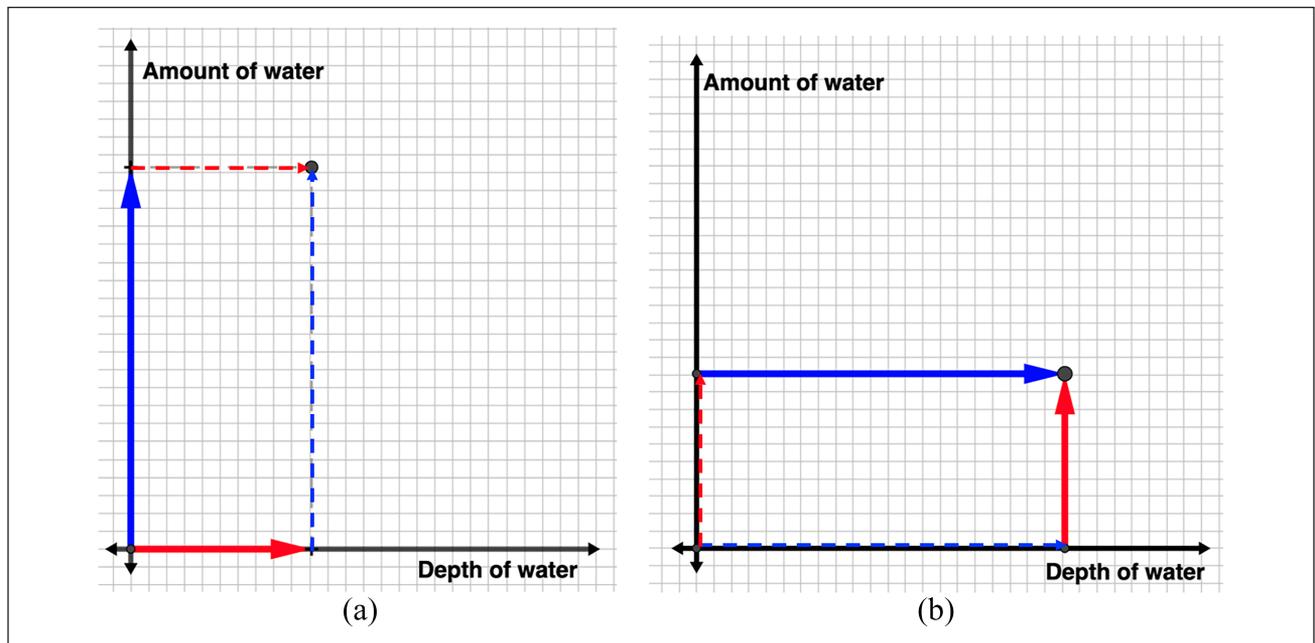
Note. In both planes, the lengths of the blue and red segments represent the magnitudes of amount of water and depth of water, respectively.

connected those points with line segments. He initially constructed Figure 14a and adjusted his graph to Figure 14b to represent bigger increments at the top half of the pool. He also drew arrows to show "increase" and "decrease" in both quantities (Figure 14a).

When the TR questioned him about his tick marks, Zane referred to each quantity's magnitude by drawing a line segment from the origin to the tick mark on the axis. He also moved his fingers on the axes to simulate the quantities' variation as the TR played the animation to fill the empty pool. After we understood that he conceived of the distance from each finger to the origin as representing the magnitude of amount and depth of water on each axis, we wanted to gain insights into the

Figure 14

(a) Zane's First Draft, (b) Zane's Second Draft, and (c) Zane Moving His Fingers on Axes
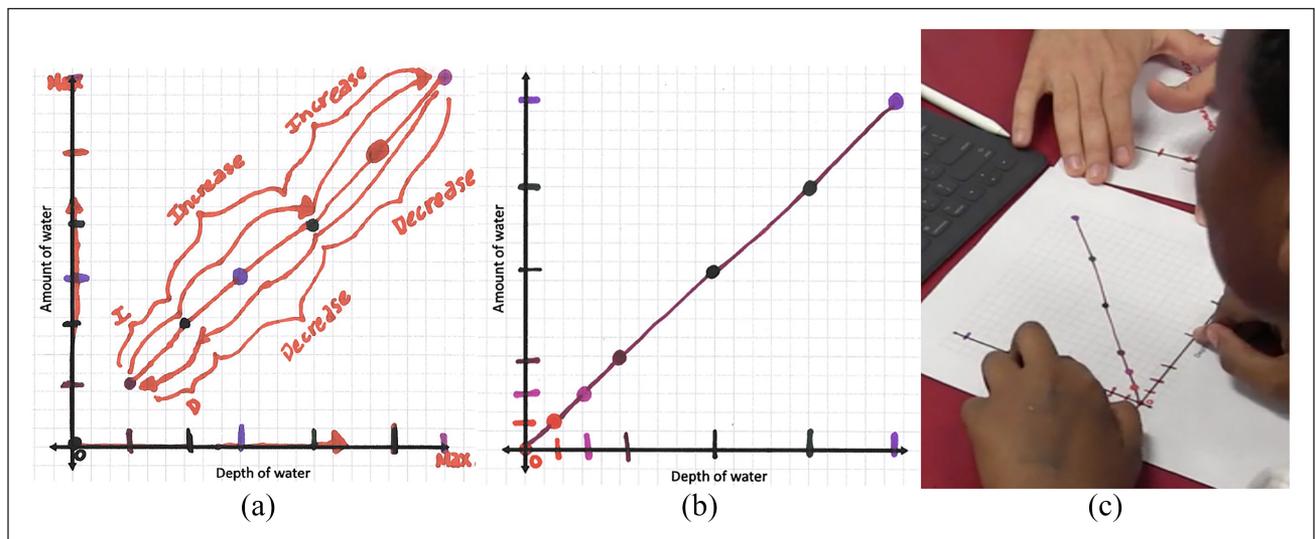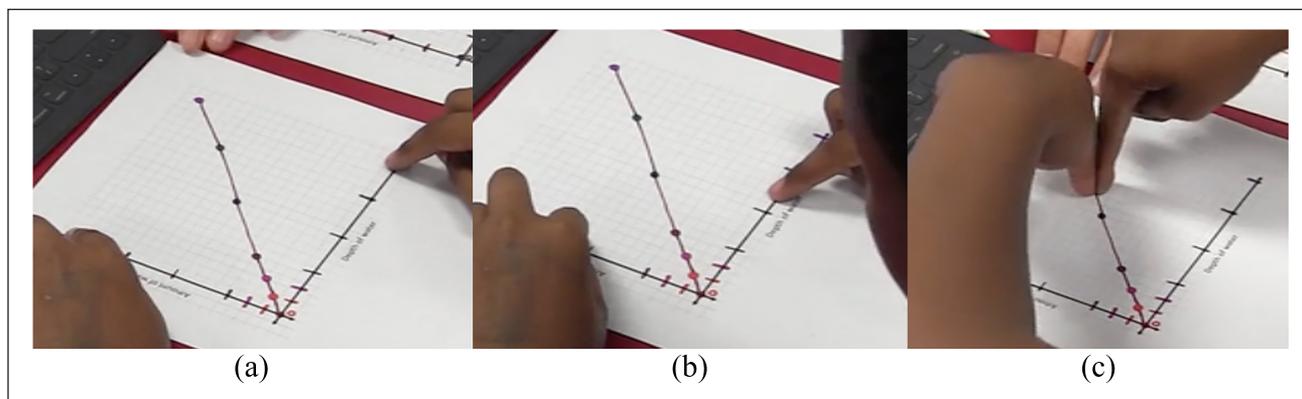
Figure 15

(a) The Location of Zane's Fingers on the Axes When the Pool Was Full, (b) the Location of Zane's Fingers on the Axes for Another State of the Pool, and (c) Zane Pointing to the Corresponding Point Where Both Fingers Met

(a)   (b)   (c)

extent to which he coordinated those tick marks on the axes to construct his points on the graph. The TR asked Zane to indicate the point on his graph that represented the amount of water and depth of water when the pool was full. Zane first simultaneously pointed to the far right and top purple tick marks on each axis (Figure 14b; also see his gesture illustrated in Figure 15a), and then he brought his fingers together simultaneously to point to the corresponding purple point on the plane (Figure 14b). His actions suggested that he could associate these two tick marks on each axis to the point on the plane. We repeated the same activity for another state of the pool situation, and Zane re-presented amount and depth of water on the axes (Figure 15b). He followed this by plotting a point on the plane where he joined his index fingers simultaneously (Figure 15c).

Taken together with his description of a dot, Zane's actions suggested that he associated the two tick marks (i.e., the indications of the quantities' magnitudes) on the axes with the corresponding point on the plane. Moreover, he held in mind two quantities associated with a point on the plane and imagined the variation of the two quantities as the point moved. That is, Zane's actions are indications that he was able to conceive of a point on the plane as a quantitative multiplicative object on a Cartesian plane that simultaneously united the two quantities' magnitudes and their variations. In fact, later in the session, he described a point moving up and down along the line as representing both quantities' increases and decreases at the same time, saying, "the dot represents both."

An Illustration From Mike's Teaching Experiment (Quantitative Multiplicative Object in a Nonconventional Cartesian Plane). To illustrate a student re-presenting a quantitative multiplicative object in a nonconventional Cartesian plane, we describe Mike's activity in the Swimming Pool Task. Note that the only difference between a student re-presenting a quantitative multiplicative object using conventional and nonconventional coordinate systems is the way the student interpreted the label of each axis as it related to how they conceived of their point as re-presenting the associated quantity's magnitude. In the Swimming Pool Task, Mike determined that "depth of water and amount of water both increase" together. To re-present this directional relationship (i.e., gross coordination of values) on a coordinate plane, he followed a series of actions outlined in Figure 16. Initially, he represented the simultaneous increase in both quantities using a straight upward line from left to right (Figure 16a). Additionally, he represented "both the depth of the water and the amount of water" for different states in the pool animation using tick marks on the graph. He also used a single point on the graph to represent each state. Furthermore, he drew vertical and horizontal segments connecting the axes to the graph (Figure 16b), with the lengths of these segments varying to reflect the magnitudes of depth of water and amount of water in the pool animation. These actions suggested that Mike's use of points stemmed from the re-presentation of quantitative multiplicative objects.

However, the way Mike organized the space was not consistent with a conventional Cartesian plane. Note that the horizontal and vertical axes in the coordinate plane we provided were labeled as "Depth of water" and "Amount of water," respectively. Contrary to the convention of a Cartesian plane, which involves an individual re-presenting the magnitude of amount of water using horizontal segments on the plane, Mike consistently re-presented the magnitude of the depth of water for different states using vertical segments. For example, when he drew a segment on the vertical side of the pool to indicate the magnitude of depth of water in the pool, he associated the same magnitude with a vertical segment in his graph (Figure 16c). Similarly, he perceived the direct distance between his graph and the vertical axis as a representation of the magnitude of amount of water. This was evident because he highlighted a horizontal segment to indicate the magnitude of amount of water in the pool (Figure 16c). Importantly, no evidence suggests that Mike considered the specific measurements of these lengths. Instead, his focus was on capturing the overall change in the magnitudes of amount and depth of

Figure 16

Sequence of Actions Mike Took When Drawing a Graph in the Swimming Pool Task

(a)   (b)   (c)
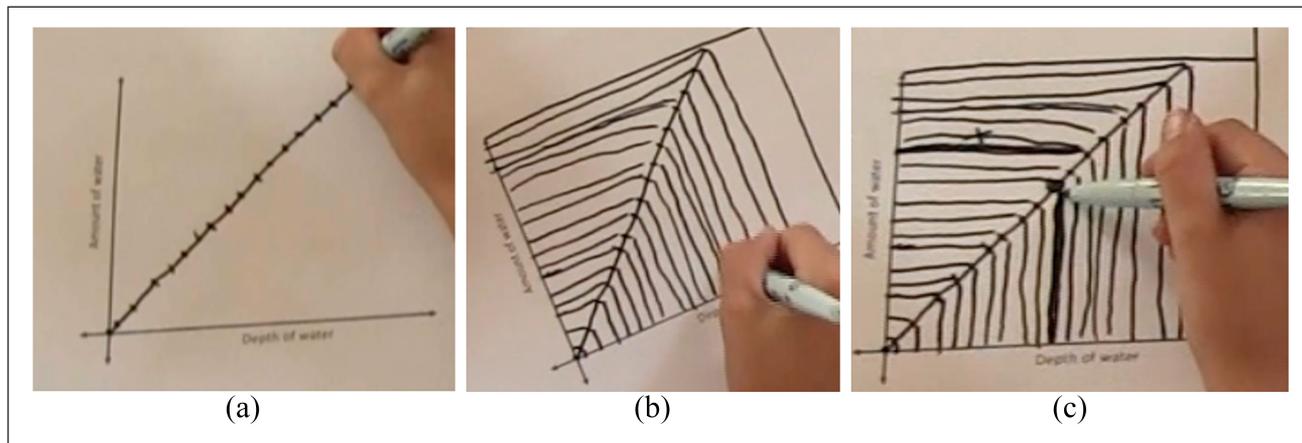
water, as depicted by the varying sizes of the vertical and horizontal segments representing these magnitudes in Figure 16b. This observation provided evidence that Mike's meaning of points included representing quantitative multiplicative objects in a nonconventional coordinate system.

Discussion

In this study, we explored middle school students' meanings of points when they graphed relationships between quantities in dynamic experiential situations. Our findings address a gap in the literature on students' understanding of points as multiplicative objects, revealing more varied and nuanced meanings than previously documented. Prior research identified iconic meanings, but we introduce new constructs, including rigidly and nonrigidly transformed iconic translations. We also differentiate novel quantitative meanings: nonunited points, spatial-quantitative multiplicative objects, and quantitative multiplicative objects in both conventional and nonconventional coordinate systems. These constructs clarify the attributes students seek to re-present when graphing relationships and provide new directions for researchers and educators supporting students' representational activities.

Our findings expand the literature on quantitative and covariational reasoning by revealing how students construct quantities and coordinate systems—often diverging from researchers' expectations. This discrepancy highlights the need for broader models that account for the variability in students' meanings. By documenting students' enacted meanings for points and their alternative uses of coordinate systems, we illustrate how spatial and quantitative reasoning interact to support emergent shape thinking. Our results also show that students construct coordinate systems through novel, creative processes, sometimes structuring the plane in ways that go beyond existing frameworks (e.g., Lee et al., 2020). Recognizing these differences between researcher-intended and student-constructed coordinate systems is a key contribution, providing insights for researchers and educators supporting diverse and robust mathematical thinking.

A key contribution of this article is the development of second-order models that can serve as foundational frameworks for further study. Hackenberg et al. (2024) described second-order models as explanatory accounts of students' learning that inform instructional design and mathematical interactions. Aligned with this perspective, our framework introduces nuanced quantitative meanings to explain how students develop emergent shape thinking in graphing. By identifying two distinct quantitative multiplicative object meanings, our study lays the groundwork for understanding broader graphing meanings that students construct.

Although this article does not explicitly focus on emergent shape thinking, that construct served as an orienting framework for the teaching experiment. Paoletti et al. (2023) suggested that conceiving of a quantitative multiplicative object is essential for students to develop emergent shape thinking as a graphical meaning. Our study identifies more nuanced quantitative meanings, including two distinct quantitative multiplicative object meanings. This nuance highlights the need to explore how these meanings relate to broader graphing understandings. Specifically, still to be examined is whether the meanings for points outlined in this study support the development of graphing meanings rooted in static or emergent shape thinking. We hypothesize that the quantitative meanings, particularly the two types of quantitative multiplicative objects, provide natural developmental foundations for emergent shape thinking, whereas iconic meanings align more with static reasoning. In the following sections, we discuss the implications of our constructs for research, teaching, and curriculum development.

Transformed Iconic Translation

We distinguished transformed iconic translation from iconic translation, which allowed a more sophisticated model of students' activities when creating or interpreting graphs. Transformed iconic translation required additional cognitive resources because students manipulated the picture or graph shape, which involved aspects of geometric reasoning. This perspective highlights students' problem-solving abilities and contrasts with the traditional view that sees iconic translation as difficult for students (Leinhardt et al., 1990). Although facilitating the development of quantitative meanings for points on the plane is important, educators must honor transformed iconic translations because they reflect students' developing reasoning and spatial sense in making sense of graphs.

Nonunited Points

Our study emphasized the complexity of constructing multiplicative objects by identifying students' use of nonunited points, which reflected their tendency to re-present individual quantities separately rather than as composite multiplicative objects. From a quantitative and covariational reasoning perspective, this may represent a developmental phase in which students recognized and graphed singular magnitudes but had not yet coordinated them into a unified structure. This hypothesis aligns with Thompson and Carlson's (2017) work on students' progressive development of covariational reasoning, which suggested that fostering connections between isolated quantities may bridge the gap to more advanced reasoning.

For example, students in our study often re-presented individual quantities to plot separate points, such as the bike's distance from two locations, without uniting the two distances so that a single point emerged from the simultaneous re-presentation of each. Although this reasoning demonstrated attention to quantitative variation, it highlighted that recognizing individual quantities alone is insufficient for forming a multiplicative object within the Cartesian plane. Importantly, students' reasoning about single quantities was inherently tied to time, whether implicitly or explicitly (Moore, 2025; Thompson & Carlson, 2017). Although we do not focus on temporal reasoning, students' interpretations of nonunited points often involved conceptual time, adding to the complexity of their reasoning. By identifying nonunited points as a key meaning, our research underscored the importance of understanding how students transition from reasoning about individual quantities to coordinating them into multiplicative objects. This insight provides a pathway for educators and researchers to support students in making this critical progression within dynamic contexts.

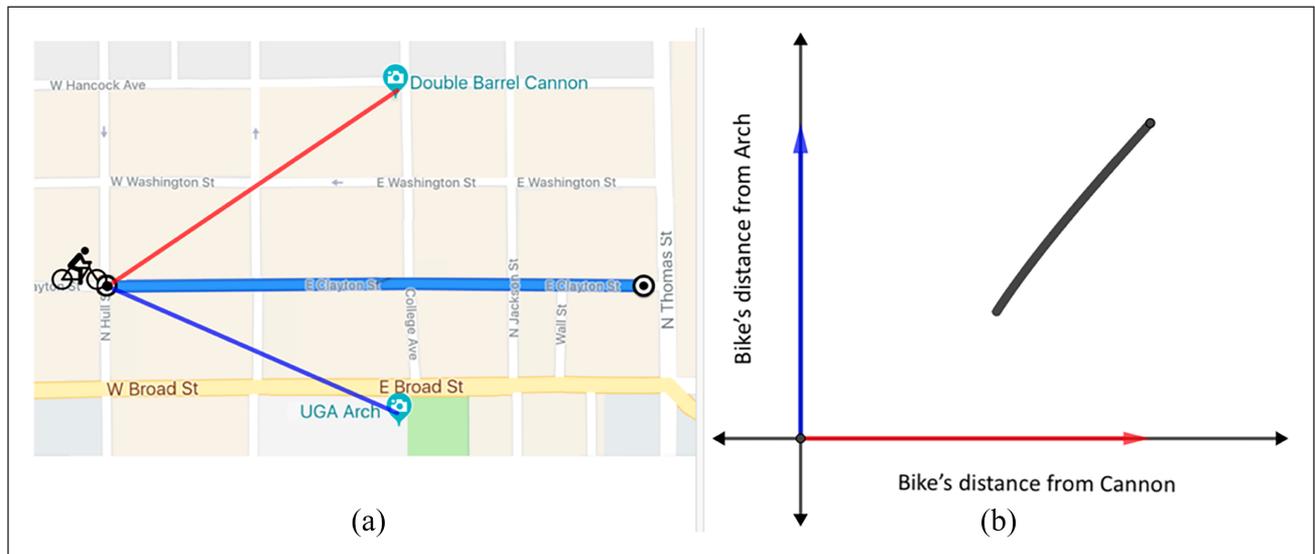**Spatial-Quantitative Multiplicative Object**

Lee et al. (2020) distinguished between spatial and quantitative coordinate systems, noting that the use of these different systems often remains implicit or conflated in curricular materials and instruction (Lee & Guajardo, 2023; Paoletti, Lee, et al., 2022). Our findings highlight the implications of such conflation, because some student reasoning exhibited characteristics of both systems. For instance, conceiving of a point as a spatial-quantitative multiplicative object involved reasoning about spatial and quantitative attributes simultaneously during re-presentation. In the Downtown Athens Bike Task, Mike re-presented a spatial-quantitative multiplicative object by isolating and disembedding quantities from the situation and projecting them onto a new space (Figure 12b), aligning with Lee et al.'s (2020) definition of a quantitative coordinate system. However, Mike also intended to replicate spatial aspects, essentially establishing both a quantitative and spatial coordinate system simultaneously. Thus, we echo Lee et al.'s (2020) call for more attention to the distinct, albeit related, uses of different coordinate systems.

Our construct of the spatial-quantitative multiplicative object challenges whether interpreting and constructing graphs are distinct processes. Graphing activities are typically categorized as either interpretation or construction (Glazer, 2011; Leinhardt et al., 1990), yet our findings from experiential contexts suggested minimal distinction between the cognitive operations involved in both. For example, drawing a moving object's path using a graph can be seen as constructing a graph, because the path is constructed through re-presenting a quantitative relationship.

To elaborate, in the Downtown Athens Bike Task (Figure 17a), the bike's distances from Arch and Cannon can be conceptualized as two magnitudes in a fixed relationship dictated by the bike's path on the map. These magnitudes can then be re-presented on a Cartesian plane (Figure 17b) while maintaining the same relationship (https://youtu.be/c_anxTI8UJQ). This activity involves the re-presentation of a quantitative multiplicative object in a Cartesian plane, and is commonly considered a constructing a graph task. Similarly, given a graph on a Cartesian plane representing the relationship between the bike's distances from Arch and Cannon (Figure 18a), we can create a corresponding path on the map by overlaying a bi-radial coordinate system (Figure 18b) and varying the magnitudes to maintain their relationship. This represents a spatial-quantitative multiplicative object, highlighting how constructing a road or path on a map can be seen as re-presenting a quantitative relationship graphically. Conceptualizing the activity in this way adds nuance to Lee et al.'s (2020) distinction by moving the real-life or graph framing to the background and foregrounding the goal-oriented nature of the activity. Producing Figure 18b involves aspects consistent with both spatial and quantitative coordinate systems

Figure 17

(a) Downtown Athens Bike Task and (b) A Graph That Represents the Bike's Distance From Arch and Cannon on a Cartesian Plane

(a) (b)

because the bike's path is constructed by disembedding quantities and their relationships from the graph and projecting them onto the map in a way that preserves them.

Quantitative Multiplicative Object in Nonconventional Coordinate Systems

Characterizing students' meanings of points as quantitative multiplicative objects in nonconventional coordinate systems places an emphasis on validating students' mathematics instead of focusing on misconceptions. For example, if a student thinking in a way similar to Mike in the Swimming Pool Task plots the point (3, 5) by going three units up from the *x*-axis

Figure 18

(a) Graph of the Bike's Distance From Arch and Cannon on Cartesian Coordinate System and (b) Two-Center Bipolar Coordinate System Overlaid Onto the Map
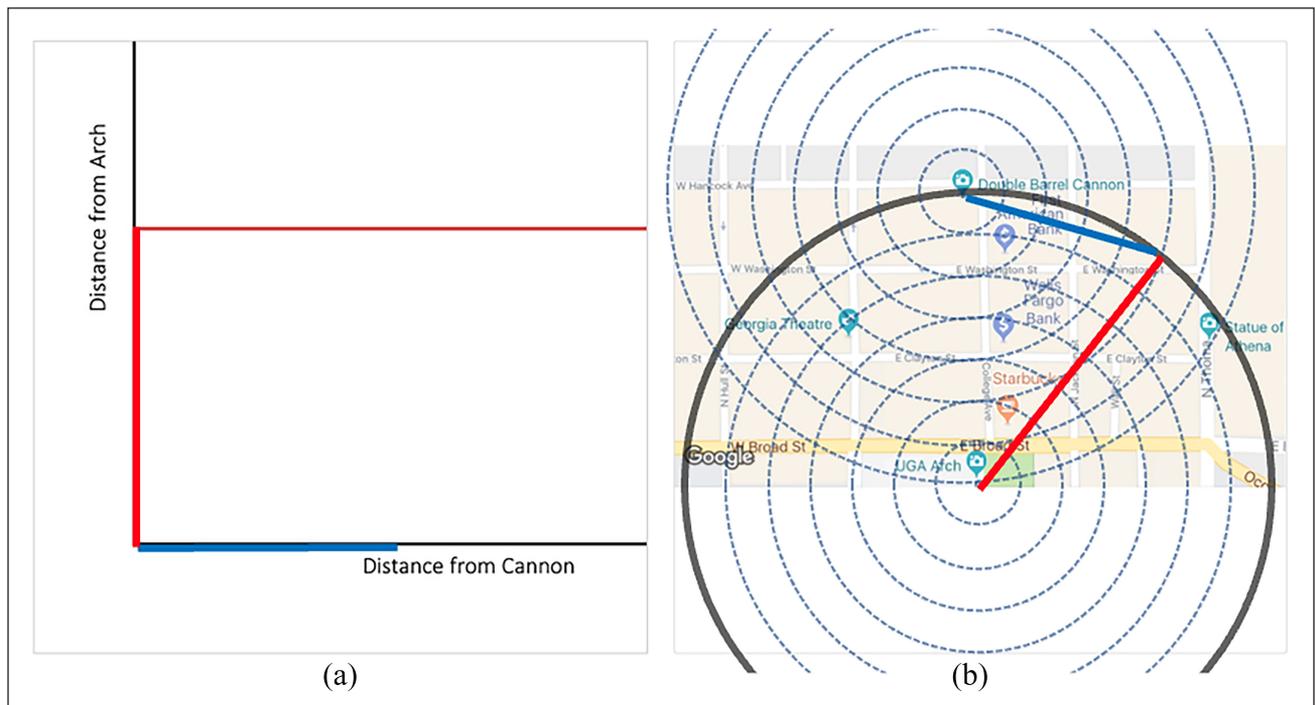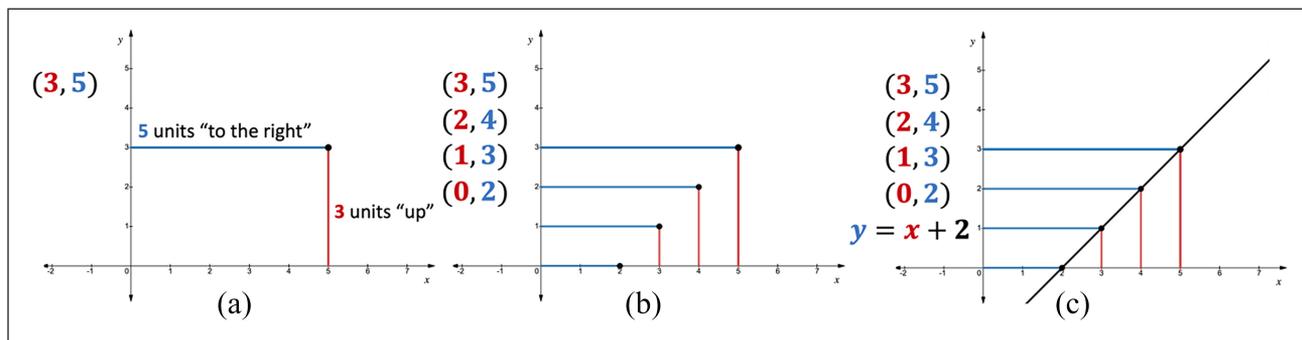

(a) (b)

Figure 19

Plotting Points and Graphing in a Nonconventional Coordinate System

and five units to the right from the *y*-axis (Figure 19a), this reasoning is logically valid and represents the student's unique understanding, but is often labeled a misconception (Sarama et al., 2003). For example, Illustrative Mathematics (2024) listed as an "anticipated misconception" that "students may confuse the order of the coordinate pairs when plotting points" (Grade 6, Unit 7, Lesson 11). However, this student can consistently plot points (Figure 19b) and create a line graph using their system (Figure 19c). Although the resulting graph may look different from one in a conventional Cartesian plane, it accurately represents the quantitative relationship between the two quantities. Labeling this as a misconception hides students' brilliance and prevents teachers and researchers from valuing their current thinking.

   A key contribution of this study is our identification of varied uses of coordinate systems, which enhances the field's understanding of students' quantitative and covariational reasoning while supporting inclusive and equitable teaching. Through this lens, we found that students constructed meaningful quantitative relationships even when their interpretations of the Cartesian plane diverged from researcher- or educator-perceived standard conventions. Rather than view a nonconventional interpretation as an error, our framework positions it as productive reasoning that can serve as a foundation for conventional graphing practices. Recognizing and valuing students' reasoning, even when unconventional, deepens our understanding of their mathematical thinking (Moore, Silverman, et al., 2014) and reveals how their conceptualizations of coordinate systems interact with representations of quantities. Our research underscores the importance of engaging with students' perspectives, acknowledging that their mathematics is sensible (Adiredja & Louie, 2020; Paoletti et al., 2024). As Hackenberg (2010a) argued, empowering students and challenging deficit thinking requires building models of their mathematics. By doing so, researchers and educators can foster flexible, robust reasoning, broadening the scope of quantitative and covariational reasoning research and practice.

Future Research Ideas

   Future research should test and refine our second-order models across diverse educational settings and student populations. Although our framework effectively explains our participants' actions, its broader applicability remains an open question. Future research should explore the usefulness of these meanings in explaining other students' mathematical realities, with iterative refinements expected as researchers encounter new cases and contexts. This process will probably lead to the extension of existing models and the development of new explanatory frameworks when students' actions cannot be fully accounted for.

   This iterative model-testing process is crucial for developing robust *epistemic subjects*—generalized second-order models that specify categorical differences among a wide range of students' meanings for a particular idea—that are useful tools for researchers and educators (Beth & Piaget, 1966; Dawkins et al., 2024; Steffe & Norton, 2014; Thompson, 2014). Through this process, researchers gain insight into developmental shifts in students' reasoning. For example, both iconic and quantitative meanings for points and graphs have been established as stable second-order models in mathematics education research. However, further study is needed to understand how students coordinate these meanings and how educators can support transitions from iconic to quantitative reasoning.

   In our teaching experiment, we designed tasks to include various types of quantities, such as static quantities (e.g., Georgia Theater's distance from Arch), lengths (e.g., bike's distance from Arch), and volumes (e.g., of water in a pool). Different types of quantities can influence cognitive activity because of their distinct representational demands. For instance, representing a line segment as a length differs cognitively from representing the volume of water in a pool, because the former is directly one dimensional whereas the latter requires transforming a three-dimensional quantity into a one-dimensional proxy when graphing. Although we observed iconic and quantitative meanings across tasks, we lack sufficient

evidence to conclude that students' understanding of these quantities shapes their meanings of points. This presents an opportunity for future research on how different quantities influence students' re-presentation activities. Additionally, the complexity of using two-dimensional representations for three-dimensional objects suggests multiple layers of cognitive processing that warrant further exploration.

Conclusion

This study offers a framework for understanding middle school students' meanings of points when constructing graphs in dynamic, experiential contexts. By identifying both iconic and quantitative meanings—including newly articulated forms like transformed iconic translations, nonunited points, and spatial-quantitative multiplicative objects—we provide researchers and educators with conceptual tools to better interpret students' graphing behaviors. These second-order models illuminate the diverse ways in which students re-present quantities and relationships, even when their representations deviate from conventional norms. Rather than view such deviations as misconceptions, our work emphasizes the value of honoring students' current thinking as a foundation for future learning. We invite further research to build on these models, explore their developmental trajectories, and apply them to design instruction that fosters deeper, more flexible reasoning about graphs and quantitative relationships.

References

Adiredja, A. P., & Louie, N. (2020). Untangling the web of deficit discourse in mathematics education. *For the Learning of Mathematics*, *40*(1), 42–46. https://flm-journal.org/Articles/7B35164CB71EB61D42002E1B2CCCFD.pdf

Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology* (W. Mays, Trans.). Reidel.

Botzer, G., & Yerushalmy, M. (2008). Embodied semiotic activities and their role in the construction of mathematical meaning of motion graphs. *International Journal of Computers for Mathematical Learning*, *13*(2), 111–134. https://doi.org/10.1007/s10758-008-9133-7

Budak, S., Sahin, N., & Dogan, B. (2018). Mathematics of motion using GeoGebra. *Mathematics Teacher*, *111*(5), 382–384. https://doi.org/10.5951/mathteacher.111.5.0382

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378. https://doi.org/10.2307/4149958

Carlson, M. P., O'Bryan, A., & Rocha, A. (2022). Instructional conventions for conceptualizing, graphing and symbolizing quantitative relationships. In G. Karagöz Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative reasoning in mathematics and science education* (pp. 221–259). Springer. https://doi.org/10.1007/978-3-031-14553-7_9

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. L. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 55–73). University of Wyoming.

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics*, *11*(1–2), 77–87.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. In P. Cobb (Ed.), *Learning mathematics: Constructivist and interactionist theories of mathematical development* (pp. 31–60). Kluwer. https://doi.org/10.1007/978-94-017-2057-1_2

Cory, B. (2010). Bouncing balls and graphing derivatives. *Mathematics Teacher*, *104*(3), 206–213. https://doi.org/10.5951/mt.104.3.0206

Czocher, J. A., Hardison, H. L., & Kularajan, S. S. (2022). A bridging study analyzing mathematical model construction through a quantities-oriented lens. *Educational Studies in Mathematics*, *111*(2), 299–321. https://doi.org/10.1007/s10649-022-10163-3

David, E. J., Roh, K. H., & Sellers, M. E. (2019). Value-thinking and location-thinking: Two ways students visualize points and think about graphs. *The Journal of Mathematical Behavior*, *54*, Article 100675. https://doi.org/10.1016/j.jmathb.2018.09.004

Dawkins, P. C., Hackenberg, A. J., & Norton, A. (Eds.). (2024). *Piaget's genetic epistemology for mathematics education research*. Springer. https://doi.org/10.1007/978-3-031-47386-9

de Freitas, E., Lerman, S., & Parks, A. N. (2017). Qualitative methods. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 159–182). National Council of Teachers of Mathematics.

Deniz, H., & Dulger, M. F. (2012). Supporting fourth graders' ability to interpret graphs through real-time graphing technology: A preliminary study. *Journal of Science Education and Technology*, *21*(6), 652–660. https://doi.org/10.1007/s10956-011-9354-8

DePeau, E. A., & Kalder, R. S. (2010). Using dynamic technology to present concepts through multiple representations. *Mathematics Teacher*, *104*(4), 268–273. https://doi.org/10.5951/mt.104.4.0268

diSessa, A. A., Hammer, D., Sherin, B., & Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. *The Journal of Mathematical Behavior*, *10*(2), 117–160.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–126). Kluwer. https://doi.org/10.1007/0-306-47203-1_7

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 215–238). Springer. https://doi.org/10.1007/978-3-642-17735-4_13

Ellis, A., Ely, R., Singleton, B., & Tasova, H. (2020). Scaling-continuous variation: Supporting students' algebraic reasoning. *Educational Studies in Mathematics*, *104*(1), 87–103. https://doi.org/10.1007/s10649-020-09951-6

Ellis, A. B., Ozgur, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An exponential growth learning trajectory: Students' emerging understanding of exponential growth through covariation. *Mathematical Thinking and Learning*, *18*(3), 151–181. https://doi.org/10.1080/10986065.2016.1183090

Ellis, A., Tasova, H. I., & Singleton, B. (2018). How quantitative reasoning can support graph understanding in algebra. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 195–198). University of South Carolina & Clemson University.

Fernández, M. L. (2006). Teaching about functions through motion in real time. *Mathematics Teacher*, *99*(6), 430–437. https://doi.org/10.5951/MT.99.6.0430

Fonger, N. L. (2019). Meaningfulness in representational fluency: An analytic lens for students' creations, interpretations, and connections. *The Journal of Mathematical Behavior*, *54*, Article 100678. https://doi.org/10.1016/j.jmathb.2018.10.003

Frank, K. M. (2016). Plotting points: Implications of "over and up" on students' covariational reasoning. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 573–580). University of Arizona.

Frank, K. M. (2017). *Examining the development of students' covariational reasoning in the context of graphing* (Publication No. 10607865) [Doctoral dissertation, Arizona State University]. ProQuest Dissertations and Theses Global.

Glazer, N. (2011). Challenges with graph interpretation: A review of the literature. *Studies in Science Education*, *47*(2), 183–210. https://doi.org/10.1080/03057267.2011.605307

González, D. A., (2021). The progression of preservice teachers' covariational reasoning as they model global warming. *The Journal of Mathematical Behavior*, *62*, Article 100859. https://doi.org/10.1016/j.jmathb.2021.100859

Hackenberg, A. J. (2010a). Mathematical caring relations in action. *Journal for Research in Mathematics Education*, *41*(3), 236–273. https://doi.org/10.5951/jresematheduc.41.3.0236

Hackenberg, A. J. (2010b). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction*, *28*(4), 383–432. https://doi.org/10.1080/07370008.2010.511565

Hackenberg, A. J., Tillema, E. S., & Gatza, A. M. (2024). Second-order models as acts of equity. In P. C. Dawkins, A. J. Hackenberg, & A. Norton (Eds.), *Piaget's genetic epistemology for mathematics education research* (pp. 475–509). Springer. https://doi.org/10.1007/978-3-031-47386-9_14

Hattikudur, S., Prather, R. W., Asquith, P., Alibali, M. W., Knuth, E. J., & Nathan, M. (2012). Constructing graphical representations: Middle schoolers' intuitions and developing knowledge about slope and *y*-intercept. *School Science and Mathematics*, *112*(4), 230–240. https://doi.org/10.1111/j.1949-8594.2012.00138.x

Illustrative Mathematics. (2024). *IM 6–8 Math* (Version 360) [Computer application]. https://illustrativemathematics.org

Inhelder, B., & Piaget, J. (1964). *The early growth of logic in the child: Classification and seriation*. Routledge. https://doi.org/10.4324/9781315009667

Izsak, A. (2000). Inscribing the winch: Mechanisms by which students develop knowledge structures for representing the physical world with algebra. *Journal of the Learning Sciences*, *9*(1), 31–74. https://doi.org/10.1207/s15327809jls0901_4

Izsák, A. (2003). "We want a statement that is always true": Criteria for good algebraic representations and the development of modeling knowledge. *Journal for Research in Mathematics Education*, *34*(3), 191–227. https://doi.org/10.2307/30034778

Izsák, A., Tillema, E., & Tunç-Pekkan, Z. (2008). Teaching and learning fraction addition on number lines. *Journal for Research in Mathematics Education*, *39*(1), 33–62.

Johnson, H. L. (2015). Together yet separate: Students' associating amounts of change in quantities involved in rate of change. *Educational Studies in Mathematics*, *89*(1), 89–110. https://doi.org/10.1007/s10649-014-9590-y

Johnson, H. L. (2022). Task design for graphs: Rethink multiple representations with variation theory. *Mathematical Thinking and Learning*, *24*(2), 91–98. https://doi.org/10.1080/10986065.2020.1824056

Johnson, H. L., McClintock, E. D., & Gardner, A. (2020). Opportunities for reasoning: Digital task design to promote students' conceptions of graphs as representing relationships between quantities. *Digital Experiences in Mathematics Education*, *6*(3), 340–366. https://doi.org/10.1007/s40751-020-00061-9

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education*, *31*(4), 500–507. https://doi.org/10.2307/749655

Koyunkaya, M. Y., Kastberg, S., Quinlan, J., Edwards, M. T., & Keiser, J. (2015). Dynamic right triangles. *Mathematics Teacher*, *109*(4), 320. https://doi.org/10.5951/mathteacher.109.4.0320

Lee, H. Y. (2017). *Students' construction of spatial coordinate systems* [Doctoral dissertation, University of Georgia]. UGA Open Scholar. https://openscholar.uga.edu/record/17006

Lee, H. Y., & Guajardo, L. (2023). A content analysis of tasks involving two-dimensional Cartesian graphs in Grade 6–8 U.S. textbooks. *Investigations in Mathematics Learning*, *15*(3), 222–240. https://doi.org/10.1080/19477503.2023.2241773

Lee, H. Y., Hardison, H. L., & Paoletti, T. (2020). Foregrounding the background: Two uses of coordinate systems. *For the Learning of Mathematics*, *40*(2), 32–37. https://flm-journal.org/Articles/79829A46DB672D1C6BA76E214A1DBA.pdf

Lee, H. Y., Moore, K. C., & Tasova, H. I. (2019). Reasoning within quantitative frames of reference: The case of Lydia. *The Journal of Mathematical Behavior*, *53*, 81–95. https://doi.org/10.1016/j.jmathb.2018.06.001

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, *60*(1), 1–64.

Liang, B., & Moore, K. C. (2021). Figurative and operative partitioning activity: Students' meanings for amounts of change in covarying quantities. *Mathematical Thinking & Learning*, *23*(4), 291–317. https://doi.org/10.1080/10986065.2020.1789930

Monk, G. S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 175–194). Mathematical Association of America.

Moore, K. C. (2014). Signals, symbols, and representational activity. In L. P. Steffe, K. C. Moore, & L. L. Hatfield (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 211–235). University of Wyoming.

Moore, K. C. (2021). Graphical shape thinking and transfer. In C. Hohensee & J. Lobato (Eds.), *Transfer of learning: Progressive perspectives for mathematics education and related fields* (pp. 145–171). Springer. https://doi.org/10.1007/978-3-030-65632-4_7

Moore, K. C. (2025). A framework for time and covariational reasoning. *The Mathematics Educator*, *33*(1). https://openjournals.libs.uga.edu/tme/article/view/3573/3536

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior*, *31*(1), 48–59. https://doi.org/10.1016/j.jmathb.2011.09.001

Moore, K. C., Liang, B., Stevens, I. E., Tasova, H. I., & Paoletti, T. (2022). Abstracted quantitative structures: Using quantitative reasoning to define concept construction. In G. Karagöz Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative reasoning in mathematics and science education* (pp. 35–69). Springer. https://doi.org/10.1007/978-3-031-14553-7_3

Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. (2014). Breaking conventions to support quantitative reasoning. *Mathematics Teacher Educator*, *2*(2), 141–157. https://doi.org/10.5951/mathteaceduc.2.2.0141

Moore, K. C., Silverman, J., Paoletti, T., Liss, D., & Musgrave, S. (2019). Conventions, habits, and U.S. teachers' meanings for graphs. *The Journal of Mathematical Behavior*, *53*, 179–195. https://doi.org/10.1016/j.jmathb.2018.08.002

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior*, *56*, Article 100692. https://doi.org/10.1016/j.jmathb.2019.01.008

Moore, K. C., Stevens, I. E., Tasova, H. I., & Liang, B. (2024). Operationalizing figurative and operative framings of thought. In P. C. Dawkins, A. J. Hackenberg, & A. Norton (Eds.), *Piaget's genetic epistemology for mathematics education research* (pp. 89–128). Springer. https://doi.org/https://doi.org/10.1007/978-3-031-47386-9_4

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. Engelke Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th annual conference on Research in Undergraduate Mathematics Education* (pp. 782–789). RUME.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of function* (pp. 69–100). Lawrence Erlbaum.

Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction*, *16*(2), 119–172. https://doi.org/10.1207/s1532690xci1602_1

Nitsch, R., Fredebohm, A., Bruder, R., Kelava, A., Naccarella, D., Leuders, T., & Wirtz, M. (2015). Students' competencies in working with functions in secondary mathematics education: Empirical examination of a competence structure model. *International Journal of Science and Mathematics Education*, *13*(3), 657–682. https://doi.org/10.1007/s10763-013-9496-7

Panorkou, N., & Germia, E. F. (2021). Integrating math and science content through covariational reasoning: The case of gravity. *Mathematical Thinking and Learning*, *23*(4), 318–343. https://doi.org/10.1080/10986065.2020.1814977

Paoletti, T., Gantt, A. L., & Corven, J. (2023). A local instruction theory for emergent graphical shape thinking: A middle school case study. *Journal for Research in Mathematics Education*, *54*(3), 202–224. https://doi.org/10.5951/jresematheduc-2021-0066

Paoletti, T., Gantt, A. L., & Vishnubhotla, M. (2022). Constructing a system of covariational relationships: Two contrasting cases. *Educational Studies in Mathematics*, *110*(3), 413–433. https://doi.org/10.1007/s10649-021-10134-0

Paoletti, T., Lee, H. Y., Rahman, Z., Vishnubhotla, M., & Basu, D. (2022). Comparing graphical representations in mathematics, science, and engineering textbooks and practitioner journals. *International Journal of Mathematical Education in Science and Technology*, *53*(7), 1815–1834. https://doi.org/10.1080/0020739X.2020.1847336

Paoletti, T., & Moore, K. C. (2017). The parametric nature of two students' covariational reasoning. *The Journal of Mathematical Behavior*, *48*, 137–151. https://doi.org/10.1016/j.jmathb.2017.08.003

Paoletti, T., Moore, K., & Vishnubhotla, M. (2024). Intellectual need, covariational reasoning, and function: Freeing the horse from the cart. *The Mathematics Educator*, *32*(1), 39–72. https://doi.org/10.63301/tme.v32i1.2854

Paoletti, T., & Vishnubhotla, M. (2022). Constructing covariational relationships and distinguishing nonlinear and linear relationships. In G. Karagöz Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative reasoning in mathematics and science education* (pp. 133–167). Springer. https://doi.org/10.1007/978-3-031-14553-7_6

Patterson, C. L., & McGraw, R. (2018). When time is an implicit variable: An investigation of students' ways of understanding graphing tasks. *Mathematical Thinking and Learning*, *20*(4), 295–323. https://doi.org/10.1080/10986065.2018.1509421

Piaget, J. (2001). *Studies in reflecting abstraction* (R. L. Campbell, Ed. & Trans.). Psychology Press.

Ponte, J. P. (1984). *Functional reasoning and the interpretation of Cartesian graphs* [Unpublished doctoral dissertation]. University of Georgia.

Radford, L., Edwards, L., & Arzarello, F. (2009). Introduction: Beyond words. *Educational Studies in Mathematics*, *70*(2), 91–95. https://doi.org/10.1007/s10649-008-9172-y

Rodriguez, J.-M. G., Bain, K., Towns, M. H., Elmgren, M., & Ho, F. M. (2019). Covariational reasoning and mathematical narratives: Investigating students' understanding of graphs in chemical kinetics. *Chemistry Education Research and Practice*, *20*(1), 107–119. https://doi.org/10.1039/C8RP00156A

Rodriguez, J.-M. G., & Jones, S. R. (2024). How students understand graphical patterns: Fine-grained, intuitive knowledge used in graphical thinking. *Journal for Research in Mathematics Education*, *55*(2), 96–118. https://doi.org/10.5951/jresematheduc-2022-0167

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. Berensen, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298–303). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Sarama, J., Clements, D. H., Swaminathan, S., McMillen, S., & González Gómez, R. M. (2003). Development of mathematical concepts of two-dimensional space in grid environments: An exploratory study. *Cognition and Instruction*, *21*(3), 285–324. https://doi.org/10.1207/s1532690xci2103_03

Singleton, B. K., & Ellis, A. B. (2020). Why multiply? Area measurement and multiplicative reasoning. *Mathematics Teacher: Learning and Teaching PK–12*, *113*(10), e37–e42. https://doi.org/10.5951/mtlt.2019.0063

Steffe, L. P., & Norton, A. (2014). Perspectives on epistemic algebraic students. In L. P. Steffe, K. C. Moore, & L. L. Hatfield (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 317–323). University of Wyoming.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Lawrence Erlbaum.

Steffe, L. P., & Ulrich, C. (2020). Constructivist teaching experiment. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 134–141). Springer. https://doi.org/10.1007/978-3-030-15789-0_32

Stevens, I. E., Paoletti, T., Moore, K. C., Liang, B., & Hardison, H. (2017). Principles for designing tasks that promote covariational reasoning. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th annual conference on Research in Undergraduate Mathematics Education* (pp. 928–936). RUME.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Sage.

Swan, M. (1985). *The language of functions and graphs: An examination module for secondary schools*. Shell Centre for Mathematical Education. https://www.mathshell.com/materials.php?item=lfg&series=tss

Tasova, H. I. (2021). *Developing middle school students' meanings for constructing graphs through reasoning quantitatively* [Doctoral dissertation, University of Georgia]. UGA Open Scholar. https://openscholar.uga.edu/record/4616

Tasova, H. I. (2022). Developing meanings for graphs from number lines: The case of Mike. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.). *Proceedings of the 44th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 331–340). Middle Tennessee State University.

Tasova, H. I., & Moore, K. C. (2021). From number lines to graphs: A middle school student's re-organization of the space. In D. Olanoff, K. Johnson., & S. Spitzer (Eds.), *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 598–602). PME-NA.

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra*. San Diego State University. https://pat-thompson.net/PDFversions/1990TheoryQuant.pdf

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). State University of New York Press.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the 32nd annual meeting of the International Group for the Psychology of Mathematics Education and the 30th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–49). Cinvestav-UMSNH.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. A. Chamberlin & L. L. Hatfield (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOMe* (pp. 33–56). University of Wyoming.

Thompson, P. W. (2014). Constructivism in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 96–102). Springer. https://doi.org/10.1007/978-94-007-4978-8_31

Thompson, P. W. (2016). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 435–461). Taylor & Francis.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, & L. L. Hatfield (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 1–24). University of Wyoming.

Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *The Journal of Mathematical Behavior*, *48*, 95–111. https://doi.org/10.1016/j.jmathb.2017.08.001

Ulrich, C., Tillema, E. S., Hackenberg, A. J., & Norton A. (2014). Constructivist model building: Empirical examples from mathematics education. *Constructivist Foundations*, *9*(3), 328–339. https://constructivist.info/9/3/328

Wavering, M. J. (1989). Logical reasoning necessary to make line graphs. *Journal of Research in Science Teaching*, *26*(5), 373–379. https://doi.org/10.1002/tea.3660260502

Van Dyke, F., & White, A. (2004). Examining students' reluctance to use graphs. *Mathematics Teacher*, *98*(2), 110–117. https://doi.org/10.5951/MT.98.2.0110

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press. https://doi.org/10.4324/9780203454220

Yerushalmy, M., & Schwartz, J. L. (1993). *Seizing the opportunity to make algebra mathematically and pedagogically interesting*. Routledge.

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