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Chapter 26

The Role of Covariational Reasoning in Understanding and Using the Function Concept

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Since 1883, there have been repeated calls for school curricula to place greater emphasis on fu tions (College Entrance Examination Board, 1959; Hamley, 1934; Klein, 1883; National Cour of Teachers of Mathematics, 1934, 1989, 2000). Despite these and other calls, students continue emerge from secondary schools and freshman college courses with a weak understanding of a cc cept that is essential for learning calculus (Carlson, 1998; Monk, 1992; Thompson, 1994a). Resear studies over the past several decades have consistently revealed that many high-performing st dents in precalculus and calculus are not conceptualizing a function as a general process that ma a set of input values to a set of output values. Complicating the matter, students encounter mu difficulty using functions to model real-world contexts that require conceptualizing quantities an how they change together (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Monk & Nemirovsk 1994; Thompson, 1994a). Understanding functions as processes that involve covarying quantiti is foundational for understanding major concepts in calculus (Carlson, Persson, & Smith, 200 Cottrill et al., 1996; Kaput, 1992; Thompson, 1994a; Zandieh, 2000). Collectively these studies ca for curriculum developers and teachers to be more attentive to how students' function conceptior are developing in relation to curriculum and teaching.

This chapter describes the processes involved in conceptualizing functions as processes tha entail two quantities varying in tandem. We make use of examples to illustrate covariational rea soning in the context of using functions to model relationships between quantities in various situa tions. We also describe a promising approach that fosters students' covariational reasoning abilitie as a primary focus for developing the function conceptions needed to understand calculus and to continue in mathematics and the sciences.

Essential Knowledge for Learning Calculus: An Overview of the Research

It has been well documented that precalculus students' function conceptions are frequently dominated by a static image of arithmetic computations used to evaluate a function at a single numerical value. One result of this constrained view, also called an *action view of function*, is that students tend to view functions only in terms of symbolic manipulations and procedural techniques disassociated from an underlying interpretation of function as a general mapping of a set of input values to a set of output values (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Dubinsky & Harel, 1992). A student with an action view of function tends to rely on computational reasoning. For the real valued function, f, defined by $f(x) = 2x^2 + 1$, students with an action view are confined to interpreting the formula as a call to execute a procedure to find a single answer for a specific value of x. They view the formula as a set of instructions to be methodically carried out: square the specific value for x, multiply this number by 2, and then add 1 to get the answer.

This chapter adapted from M. Carlson, S. Jacobs, E. Coe, S. Larsen, & E. Hsu (2002), Applying covariational reasoning while modeling dynamic events: A framework and a study, *Journal for Research in Mathematics Education*, *33*, 352–378.

While a student with an action view of function focuses on executing calculations, a student with a *process view of function* sees $f(x) = 2x^2 + 1$ as a mapping of *any* input value of the function represented by x to an output value represented by $f(x)$. Such a student does not require executing calculations to reason about the mapping; the student understands that the function f defines a mapping regardless if he or she performs the calculations associated with this mapping. Because students with a process view are not tied to executing calculations, they also view a graph as a representation of how the values of x and $f(x)$ change together—e.g., as x increases, $f(x)$ increases by greater and greater amounts. When students no longer have to imagine each individual operation for an algebraically defined function, they are able to imagine a continuum of input values in the domain of the function corresponding to a continuum of output values all at once. In one study (Carlson, 1998), 43 percent of college algebra students attempted to evaluate $f(x + a)$, given that f is defined by the formula $f(x) = 3x^2 + 2x - 4$, by adding a onto the end of $3x^2 + 2x - 4$. When explaining their thinking, the students typically argued that when adding a to one side of the equation you must also add a to the other side. As opposed to understanding that $f(x + a)$ represents the output of the function for an input of $x + a$ (e.g., a mapping between an input quantity and output quantity), the students were focused on the formula and notation as a call to execute specific calculations.

With a process view of function, a student is not tied to specific values and executing calculations and is thus posited to reason about varying quantities. The ability to interpret how the output values of a function are changing while imagining changes in a function's input values is essential for modeling a dynamic situation. This ability has been referred to as *covariational reasoning* (Carlson, 1998; Thompson, 1994a). An essential and often overlooked first step in modeling covarying relationships is conceptualizing the relevant quantities in a problem context (e.g., length of the radius and volume of an expanding sphere; angle measure and vertical distance). It is only after the relevant quantities in a word problem have been conceptualized that students are able to think about how the quantities are related and how they change together. As a clarifying example, imagine watching runners in a race. One may initially notice a starting line, and then when the starting gun is fired be interested in how the distance of a runner from the *finish line* changes as the runner is moving down the track. One might also notice that the elapsed time since the runner started the race is increasing and that the total length of the race is 100 meters. The observer has conceptualized two varying quantities, the elapsed time since the start of the race and the distance of the runner from the finish line, and one fixed quantity, the total length of the race. Conceptualizing the quantities involves identifying the measurable attributes of objects in the situation (Smith & Thompson, 2007). Once the quantities have been identified, function formulas and graphs are constructed to illustrate how the quantities change together. Students are more likely to define meaningful formulas and graphs if they are required to be specific when defining variables by saying exactly what quantity's value the variable is representing (e.g., d = the distance measured in meters of the runner from the finish line since the race began) (Moore & Carlson, 2012).

Covariational reasoning involves gaining clarity about *how* the runner's distance from the finish line is changing with the elapsed time since the runner started the race. This could involve considering fixed amounts of change of time since the race began (e.g., $^1/_2$ second) while considering how the distance of the runner from the finish line changes with each successive $^1/_2$-second increment of time. If we were to observe that the runner was traveling a greater distance for each successive $^1/_2$-second increment since starting the race, we might conclude that the runner's distance from the finish line is decreasing at an increasing rate of change over that interval of time. In the following section, we elaborate on what is involved in covariational reasoning and illustrate why this approach to representing and interpreting function relationships is so powerful for students.

Covariational Reasoning: A Powerful and Foundational Way of Thinking

The Bottle problem (see fig. 26.1) provides a rich context for both developing and assessing students' covariational reasoning abilities in terms of several ways of thinking about covarying quantities (see table 26.1). In this problem students are asked to construct a rough sketch of a graph of a function that defines the height of water in a bottle in terms of (or as a function of) the amount of water in the bottle.

Imagine this bottle filling with water. Sketch a graph of the water's height in the bottle as a function of the amount of water in the bottle. Explain the thinking you used to construct your graph.

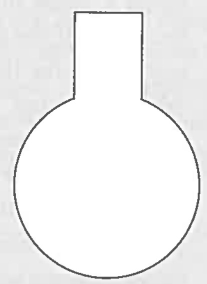

Fig. 26.1. The Bottle problem

Table 26.1
Mental actions of the Covariation Framework (from Carlson et al., 2002)

Mental action	Description of mental action	Behaviors (graphical representation)
Mental Action 1 (MA1)	Coordinating the dependence of one variable on another variable	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing a monotonic straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate of change of the function with uniform increments of change in the input variable	• Constructing secant lines for contiguous intervals in the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the changes in the instantaneous rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

When responding to this problem, high-performing college algebra students at the end of their (e.g., "the height goes up") while imagining increases in the amount of water (table 26.1, MA2). Most second-semester calculus students in the study constructed a concave up graph. Analysis of the interview data revealed that these students were imagining the water getting higher and higher in the bottle. To them, "higher and higher" in reference to the water in the bottle meant "steeper and steeper" on the graph. They were not covarying the amount of volume of water with the height of the water in the bottle, leading to an incorrect graph. Other college algebra students constructed a strictly concave down graph and provided a justification indicating that they were trying to construct a shape that matched the side of the bottle (Carlson, 1998). This type of error in which the shape of the graph is made to match some physical attribute of the motion or event is widely reported in the literature (Monk, 1992; Thompson, 1994b) and has been referred to as one form of "shape thinking" (Thompson, 1995). Our continued research has revealed that when students are supported in conceptualizing the quantities (amount of water and height of water in the bottle) and how they change together, they do not engage in "shape thinking." Rather, they are able to think about points on the graph as representations of amounts of water and height of water, and they can easily move from one point on a graph to another by imagining how the height of the water in the bottle changes with incremental increases in the amount of water in the bottle.

Most students who gave the correct response (a concave down graph, followed by an inflection point where the graph becomes concave up, followed by a linear portion in which the slope of the line is the same as the slope of the tangent line at the point it becomes linear) gave explanations that revealed they were imagining adding successive and equal small amounts (e.g., small cups) of water while thinking about the amount of increase in the height of the water in the bottle (table 26.1, MA3). Students who engaged in such reasoning correctly constructed a graph that is initially concave down and then becomes concave up at the point that corresponds to the amount of water needed to fill the bottle to the widest point on the sphere. Other students engaged in covariational reasoning by imagining how the amount of water would change when increasing the height by some fixed amount; yet other students focused on how the average rate of change of the height with respect to volume changed, while imagining fixed increases in the volume (MA4). We have formalized these ways of thinking in what we call the *Covariation Framework* (table 26.1) to illustrate ways of thinking that are helpful to develop in students.

Ideally we want students to be able to engage in all mental actions described in the covariation framework. It is particularly important that they be able to move fluidly between MA3 and MA4/5 as needed to justify why, say, the height of the water with respect to the amount of water in the bottle is increasing at a decreasing rate. We would want students to reason that, as the amount of water in the bottle varies from no water in the bottle until the bottle is half full, the height of the water in the bottle is increasing and the amount of change of the height is decreasing when continuing to add fixed amounts of water. This explanation demonstrates an ability to move from MA4/5 to MA3 by unpacking the meaning of increasing at a decreasing rate. The above Bottle problem narrative is rich in quantities and comparing quantities (e.g., amounts of change). Our research has further revealed that it is critical that students first conceptualize the quantities before being asked to covary the quantities to create graphs that capture this covariation (Moore & Carlson, 2012). This is much more likely to lead to productive reasoning and the construction of a meaningful and accurate graph.

It is natural for students to start by imaging the direction of change of two quantities (e.g., as the distance a car has been traveling since leaving home increases, the distance of the car from its destination decreases). However, a more accurate representation of how the quantities change together requires that students think systematically about how much one quantity is changing while considering a fixed amount of change in the other quantity. We now elaborate on how this knowledge (of productive ways of thinking that are involved in covariational reasoning) has informed our curriculum development (Carlson, Oehrtman, & Moore, 2013) and instructional design.

An Instructional Activity Using the Bottle Problem to Introduce Covariational Reasoning

Instructional activities designed to develop students' covariational reasoning abilities are more productive for students if the activity moves carefully through several phases to assure that all students' minds are engaged in constructing productive images of the quantities and their relationships. The following phases focus on eliciting specific ways of thinking that are developmentally necessary to construct meaningful formulas and graphs. The phases involve: (i) conceptualizing the varying quantities in the situation; (ii) imagining the direction of change of the two quantities as they vary together; (iii) thinking about how much one quantity changes while thinking about fixed incremental changes in the other quantity; and (iv) thinking about how the rate of change of one quantity (with respect to the other quantity) changes. We present the phases in the order we suggest for probing students about their graph. However, as noted above, our data has revealed that students do not always follow this order in their thinking when prompted to construct a graph.

Instructional phases for the Bottle problem

Phase I: *Conceptualizing the varying quantities*

To support students in first conceptualizing the varying quantities, the instructor brings a bottle and a pitcher of water into class. As the teacher pours the water from the pitcher into the bottle, she reminds students of how to identify and think about quantities in a situation by asking them to identify the attributes of the bottle situation that can be measured. Even though this may seem obvious to us, it has been consistently documented that students' inability to focus on quantities in a problem context is a primary reason why they are unable to construct meaningful formulas or graphs to model situations. If students say a quantity is the bottle (a common response) they are clearly not thinking about an amount of something. It is useful to remind students that graphs and formulas are constructed to illustrate how the values of two quantities change together. Acceptable responses to the task of identifying quantities include the volume of water in the bottle, the height of the water in the bottle, the surface area of the bottle, the height of the bottle, the volume of the bottle, the amount of time since starting to pour, etc. It is also useful to ask the students to distinguish between quantities with varying values and those with constant values.

Phase II: *Conceptualizing the direction of change of the two quantities as they change together*

This instructional phase most often leads to students engaging in directional covariation (MA2). After identifying the varying and nonvarying quantities in the problem context, the teacher prompts students to focus on two varying quantities only, and to imagine how they are changing together. Since the two varying quantities that are the focus of the intended activity are the volume of water in the bottle and the height of the water in the bottle, the teacher focuses the conversation on these two varying quantities by prompting students to describe how the height of the water in the bottle and the amount of water in the bottle change together.

Phase III: *Thinking about and representing how two quantities change together*

When first being introduced to reasoning about quantities and how they change together, students benefit by directly covarying the quantities in a real context or by using an animation to simulate the covariation of the quantities. After the first two phases, we suggest breaking students in groups and giving each student a spherical bottle (or several bottles), a container of water, and an object (tablespoon, eye dropper, small cup) to explore how the volume of water in the bottle and the height of the water in the bottle change together. Students perform this experiment in groups, and each student is required to discuss and then individually produce a written response to the following prompts.

- Use the given container of water, empty cup, ruler, and spherical bottle to make a rough sketch of a graph that represents how the *height of water in the bottle* covaries with the *amount of water that is in the bottle.*
- Describe the thinking you used to sketch your graph.
- What does your graph convey about how the *height of the water in the bottle* changes as the *volume of the water in the bottle* increases from empty to full?

As the students work on the activity, we suggest that the teacher monitor the students and ask them to explain the reasoning they used to construct their graphs. The students should engage with each other to compare their graphs and discuss their reasoning.

Phase IV: *Conceptualizing how the rate of change of one quantity with respect to the first quantity changes for fixed incremental changes in the first quantity*

In this phase, students explore how the *rate of change of the height with respect to the amount of water* changes over the domain of the function (as water in the bottle varies from no water to the amount of water that fills the bottle). It is noteworthy that some students talk about the water filling faster where the graph is steeper, and filling slower where the graph is less steep. This type of explanation fails to explain how the quantities are changing together and does not demonstrate that students see the graph as a representation of the covariation of *two* quantities. If students provide such explanations, ask them how they know that a steeper graph implies that the water is filling "faster" (note that time is not represented on the graph). It is productive if students are able to move from explaining the rate of change of the function over intervals of the domain to describing how the *amounts of change of the two quantities are changing together* on that same interval. For instance, if the students explain that the height is increasing at an increasing rate, they should explain that for each successive cup of water added, the height increases by more and more. This level of meaning is productive for constructing and interpreting models of dynamic events. It also lays the groundwork for understanding the idea of derivative and rate of change functions in calculus. Guiding questions for this conversation can be:

- Over what intervals of the domain is the *rate of change of the height of the water in the bottle with respect to the volume of water in the bottle* increasing at an increasing rate?
- Identify the interval on your graph over which the *rate of change of the height of water with respect to the volume of water in the bottle* is increasing. Now, explain how the *height of the water in the bottle* is changing with fixed increases in the *volume of water* on this same interval. Illustrate this on your graph and a diagram of your bottle.

We again note that these phases may not occur in a linear fashion. For instance, during phase III, students may discuss rates of change when creating their graph or they may immediately move to discuss the amounts of change of the two quantities. Or students may merely explain the directional covariational of the two quantities. It is not necessarily important that their covariational reasoning emerge in a particular order, but instead that they connect each of the mental actions. In the case that the students only describe the directional covariation for their graph, a teacher can engender a focus on other ways of thinking by proposing multiple graphs that have the same directional covariation but convey different rates of change (e.g., a linear graph or a graph with the opposite concavity). To better illustrate such a process, we describe covariational reasoning and modeling circular motion in the next section.

Developing Students' Covariational Reasoning Abilities in the Context of Circular Motion

Another context that supports modeling through covariational reasoning is that of circular motion. Modeling the motion of an object traveling in a circular manner is central to the study of trigonometric functions, but considering how quantities covary in such a situation need not be delayed until the introduction of trigonometric functions. Trigonometric functions like *sine* and *cosine* merely provide formal names and notation for functions, but the use of these names is not necessary for students to reason about the covariational relationships that these functions define.

Numerous situations involve circular motion. These include a person taking a Ferris wheel ride, a satellite orbiting the Earth, and a ball on a string rotating about a fixed point. To illustrate covariational reasoning, we will consider a person taking a Ferris wheel ride. Let's begin by imagining a person boarding a Ferris wheel from a platform at the bottom of the ride (fig. 26.2).

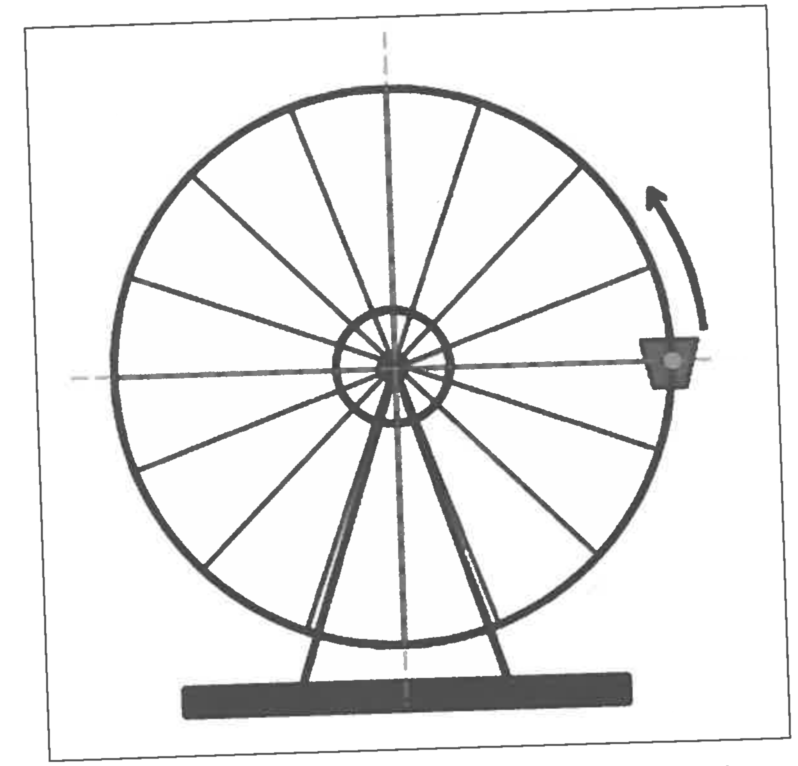

Fig. 26.2. Diagram of the Ferris wheel ride where the rider boards from the bottom

When attempting to model a situation using graphs, students frequently attempt to recall or create graphs before developing a robust understanding of the situation. As mentioned in the Bottle problem, it is important that students first conceptualize and identify quantities in the situation, including those with varying and nonvarying values. The rider's perpendicular distance from the ground, perpendicular distance from the platform, and distance traveled along the arc traced out by his trip form several quantities with varying values. The rider's distance from the center of the Ferris wheel, the diameter of the Ferris wheel, and the height of the platform off the ground form several quantities with nonvarying values. Two quantities that a student can covary in order to model the rider's trip around the Ferris wheel are the rider's perpendicular distance from the ground (*distance from the ground*) and the distance the rider has traveled (*distance traveled*) along the arc traced out by his trip.

Determining the relationship between the rider's distance from the ground and distance traveled requires that students engage in several of the mental actions identified in table 26.1. First, a student can identify that the distance from the ground is increasing as the distance traveled increases to the point at which the rider is at the top of the Ferris wheel. Then, the rider's distance from the ground decreases as the distance traveled increases to the point at which the rider is at the bottom of the Ferris wheel (MA2). From here, the vertical distance from the ground repeats the same cycle as the distance traveled continues to increase.

As with the Bottle problem, our work with students suggests that they encounter little difficulty determining the directional covariation, and it is not uncommon for students to subsequently sketch a curve that resembles the correct graph. But when pressed to justify their graph, students encounter difficulty in explaining the shape of their graph in terms of covarying quantities. Instead, they might explain the curvature as stemming from the Ferris wheel being a circle (e.g., a curve produces a curve) and a "continuous" or "smooth" ride. While it is true that the shape of the Ferris wheel influences the covariational relationship, such a justification is consistent with "shape thinking" and circumvents a focus on two quantities and how they vary in tandem. In the case that students do discuss *rates of change* between quantities (e.g., for an increasing distance traveled, the vertical distance from the ground increases at an increasing rate), we have rarely observed students unpacking such a relationship in terms of *amounts of change* between the two quantities and illustrating these *amounts of change* on both the graph and situation diagram.

One approach to foster and connect reasoning about *rates of change* and *amounts of change* is to propose different graphs that fit the appropriate *directional covariation* but are such that the graphs convey different *amounts of change* and *rates of change* information. For instance, both graphs in figure 26.3 represent the same *directional covariation* (e.g., for an increasing distance traveled, the distance from the ground increases, decreases, and then increases). Yet, these graphs differ in that the curved dotted gray graph shows the distance from the ground varying at a nonconstant rate while the straight black graph shows the distance from the ground varying at a constant rate for each interval of increase and decrease. By posing a collection of such graphs to students and tasking them with determining the appropriate graph for the situation, students are required to investigate how the quantities covary and most find it helpful to use a diagram of the situation to accomplish this.

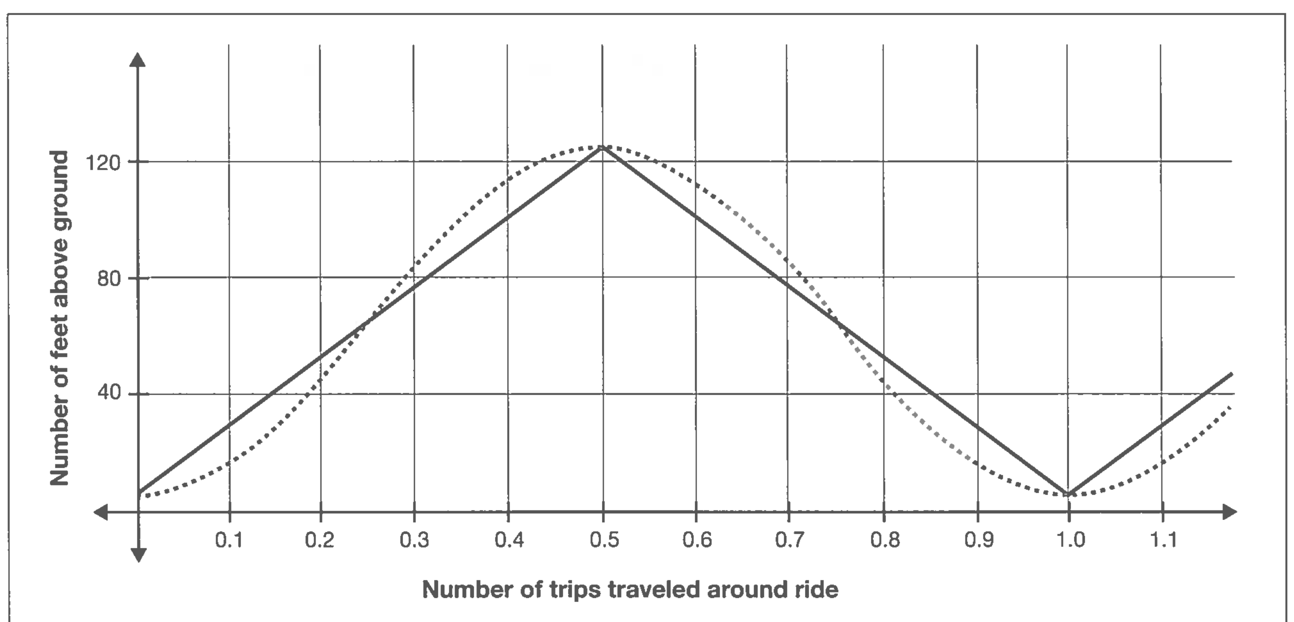

Fig. 26.3. Two graphs that convey the same *directional covariation*

A systematic approach to considering how the quantities covary is to compare changes in the distance from the ground for equal successive increases in the distance traveled. Doing so for the first quarter of a trip around the ride, a student can reason that for each successive increase in the distance traveled, the distance from the ground increases by greater and greater amounts (table 26.1, MA3). That is, the *amount of change* in the distance from the ground increases for successive equal *amounts of change* of distance traveled (fig. 26.4). If the student considers the total distance traveled increasing continuously over the first quarter of the trip, the student can conclude that the distance from the ground is increasing at an increasing rate.

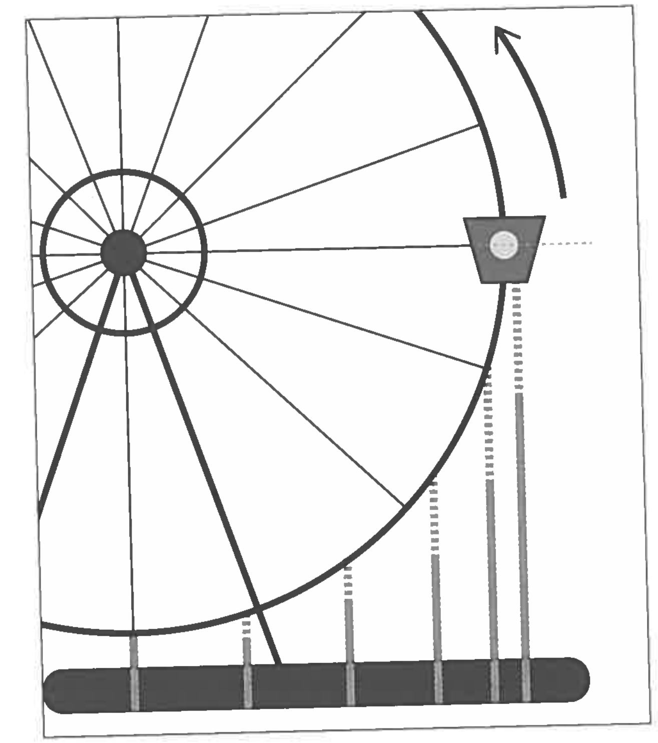

Fig. 26.4. Using a diagram to illustrate amounts of change

Research has documented students' difficulties when attempting to reasoning about how two *amounts of change* are changing together (MA3), and thus a teacher should direct the students to identify and describe the amounts of change as they are illustrated on both their graphs and diagrams of the Ferris wheel. It is also helpful to have students identify the *change in the distance of the rider from the ground* and the change in the distance that the rider has traveled as covarying quantities. Students should identify the *change* in the distance from the ground as stemming from comparing two distances from the ground, a comparison that can be facilitated by the use of the diagram as shown in figure 26.4. In this figure, the covariation is denoted by partitioning the arc length into equal intervals, identifying the distance from the ground at each partition (i.e., the sum of the solid gray and dotted gray segments), and determining the change from the previous distance from the ground (i.e., the dotted gray segment). In doing so, we observe that the change in the height of a bucket from the ground is getting greater and greater over the first quarter of a rotation.

With a more robust understanding of the covariational relationship between the distance from the ground and the distance traveled, a student can create a graph with the correct concavity. By sketching a graph such that the distance from the ground increases by greater and greater amounts for equal successive changes in the total distance, students can obtain the portion of the graph presented in figure 26.5, which captures the same covariational relationship as figure 26.4.

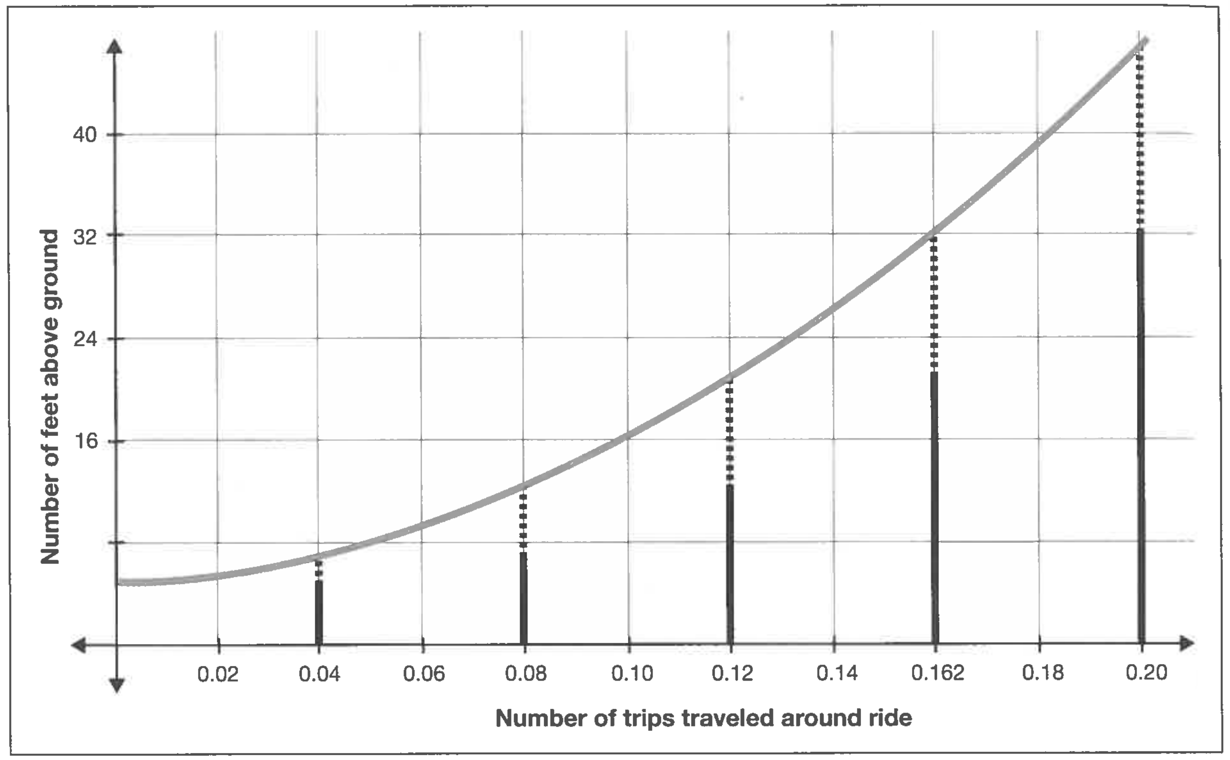

Fig. 26.4. Using a diagram to illustrate amounts of change

In figure 26.5, the covariation is denoted by partitioning the horizontal axis (i.e., distance traveled) into equal intervals, identifying the distance from the ground at each partition (i.e., the sum of the black and dotted black segments), and determining the change from the previous distance from the ground (i.e., the dotted black segment). Next, by considering the covariational relationship over the last three-quarters of a trip around the Ferris wheel, the students determine the relationship represented by the dotted gray graph in figure 26.3.

Just as students' activity of filling a bottle should lead to their volume-height graph for the Bottle problem, a central focus of the above discussion is using the Ferris wheel situation to construct a covariational relationship that can then be represented using a graph. As opposed to applying a "known" function to obtain a graph, and then using the graph to investigate the covariational relationship, students' understanding of how the quantities covary within the situation should inform the construction of their graph; students should "see" a graph as emerging from conceptualizing the quantities and how they change together, instead of from connecting points on a graph. To support students' modeling abilities, it's also important that they consider the covarying quantities "slowly" from the beginning of the ride. It is often the case that students try to picture the entire graph all at once, an action that stems from believing modeling to be fitting previously known graphs or formulas to a situation. Thus, it is important to draw their attention to considering covarying quantities over small intervals, as we have illustrated above.

Concluding Remarks

Students should come to see modeling as a generative activity; modeling should be an activity in which they come to view graphs and formulas as stemming from and representing their understanding of the covarying quantities. Developing this approach to modeling requires giving students repeated opportunities to engage in covariational reasoning so that they can make sense of situations and *then* produce graphs that reflect covariational relationships. The above situations can be modified in several ways so that the students can produce graphs that represent different relationships, but all of which foreground covariational reasoning. Variations in the Bottle problem include having them use different axes orientations (e.g., volume of water on the vertical axis in one graph and then volume of water on the horizontal axis on a second graph) and different bottles. Variations in the circular motion task include using different starting positions and different rotational directions (note that the rotational direction might not influence the graph, depending on the quantities under consideration). Other extensions can prompt the students to focus on other varying quantities, such as the perpendicular distance of the rider from the top of the Ferris wheel or the perpendicular distance of the rider from the center of the Ferris wheel. In each case on each task, students can use covariational reasoning to determine and graphically represent a dynamic relationship between the quantities.

We focus extensively on graphing in this chapter because graphs provide a productive way to represent covarying quantities. But it is just as important that students come to understand formulas as emergent representations of covarying relationships. In our recent research (Moore & Carlson, 2012) we have found that precalculus level students have not been adequately supported in identifying and conceptualizing quantities in word problem contexts. It is common for students to write d = distance and t = time without thinking about what distance or what time is being represented. If they have not first conceptualized the quantities whose values are being represented by the variables, it is impossible for them to construct a meaningful formula to relate the two quantities. As another example, we asked students to express the length of the base of a box that is formed by cutting square cutouts from each corner of an 11-inch by 8.5-inch sheet of paper. When responding to this task it was common for students to write l = length, and then to confuse the length of the base of the box with the original length of the paper. This resulted in some students expressing the length of the base of the box as $2x - 11$ or $x - 11$, and some students constructing no image of a varying base to the box and simply writing 11 for the length of the box. These students were clearly not thinking about how the length of the base of the box was changing with the length of the cutout. Students were only able to construct the correct formula, $l = 11 - 2x$, after conceptualizing a box with a varying base, defining some variable (e.g., x) to represent the length in inches that is cut from each end, and l to represent the length of the base of the box formed from cutting equal amounts from both ends of an 11-inch-wide piece of paper. It is this imagery, that of a dynamic situation composed of covarying quantities, that allows students to develop meaningful formulas (and graphs) from applied contexts. It is much too common for these critical reasoning abilities to be overlooked in curriculum and instruction, resulting in teachers showing and students searching for procedures for attaining answers.

We have leveraged the products of this and other research about ideas of precalculus level mathematics to design and refine teacher workshops, student curriculum, and instructional support tools (Carlson, Oehrtman, & Moore, 2013). The shifts we have documented in student learning in this context are very positive and highlight the value of design research to inform the development and refinement of models to support improvements in precalculus teaching and student learning. We are eager to scale this model to other sites and welcome inquiries from schools and districts that want to learn more.

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