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Complexities in students' construction of the polar coordinate system

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ABSTRACT

Despite the importance of the polar coordinate system (PCS) to students' study of mathematics and science, there is a limited body of research that explores students' ways of thinking about the PCS. Research on students' construction of the PCS is especially sparse. In this article, we highlight several issues that arose spontaneously during a teaching experiment that explored students' construction of the PCS. We illustrate how students' angle measure meanings influenced their construction of the PCS. We also discuss how the students' ways of thinking about the Cartesian coordinate system (CCS) became problematic as they transitioned to the PCS. Collectively, we highlight that students' ways of thinking about coordinate systems evolve when students reason within and across multiple coordinate systems.

1. Introduction

The *polar coordinate system* (*PCS*) has a multitude of real-world uses, a significant number of which exist in physics, engineering, and other STEM fields. In mathematics, students typically encounter polar coordinates for the first time in pre-calculus, with an emphasis on conversions between Cartesian and polar coordinates and recognizing 'special' graphs (e.g., limacons and cardioids) (Montiel, Vidakovic, & Kabael, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009). Calculus students use integration to determine areas bounded by polar curves. In multi-variable calculus, students extend their use of polar coordinates to spherical and cylindrical coordinates in order to model three-dimensional objects and relationships.

Several researchers (Montiel et al., 2008; Sayre & Wittman, 2007) identified that students rely on procedures in these courses or related situations. Moreover, students' procedural treatment of the PCS allows them to overlook various conventions and features of the PCS that differ from the *Cartesian coordinate system* (*CCS*). We thus see a need to investigate how students construct and coordinate various quantities and conventions associated with the PCS. Responding to this need, we conducted a teaching experiment (Steffe & Thompson, 2000) focused on the PCS with a group of undergraduate students enrolled in a course for pre-service secondary mathematics teachers. In this article, we describe sources of students' cognitive disequilibrium (or perturbation) that occurred during the teaching experiment. One source of student difficulty was their meanings for radian angle measure. Other sources of student difficulty stemmed from the students taking what we take to be CCS conventions (e.g., coordinate ordering as (input, output)) as necessary features of any coordinate system.

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In what follows, we first discuss the theoretical framing of the study and prior research on the PCS. We summarize the design principles of the instructional activities associated with the teaching experiment and describe our methods. We then identify three sources of cognitive disequilibrium for students in the teaching experiment and characterize the students' ways of thinking that led to these states of disequilibrium. We conclude by connecting our findings to broader areas of research and we provide ideas for future research.

2. Theoretical framing

We draw on radical constructivist theories of knowing (von Glasersfeld, 1995), adopting the stance that an individual's knowledge is fundamentally unknowable to someone other than that individual. We explain and model students' knowledge using Thompson and Harel's (Thompson, Carlson, Byerley, & Hatfield, 2014) system of *understanding*, *meaning*, and *ways of thinking*, which has roots in Piagetian notions of mental action, scheme, assimilation, and accommodation. Most relevant to the present work, we say a student has an understanding when he or she reaches a cognitive state of equilibrium by assimilating a situation to a scheme (Skemp, 1962, 1971; Thompson, 2013). According to Thompson and Harel (Thompson et al., 2014), a student's meaning or way of thinking is the space of implications that results from assimilation or a state of understanding. A student has a way of thinking about a concept or idea when the student develops a meaning that he or she uses habitually to reason about that concept or idea (Thompson et al., 2014). For instance, we assumed that the students in the present study had repeatedly constructed particular meanings for the CCS and related concepts (e.g., function) through their prior coursework and hence had established ways of thinking about the CCS and related concepts (e.g., function means execute the vertical line test). As such, our interest in this study included determining how students' ways of thinking about the CCS influence their construction of the PCS and identifying the complexities these ways of thinking create. We use the term *construction* to refer to the mental actions, operations, and abstractions entailed in a student establishing a meaning or way of thinking for some concept or intertwined concepts (e.g., a coordinate system, function, and angle measure). We use the term *complexity* (or *complexities*) to mean an instance (or instances) in which a student experiences some sense of cognitive disequilibrium that results from her or his attempts to assimilate some experience or situation to a meaning or way of thinking.

3. Background

Available research on student thinking in PCS contexts is scarce, and the extant literature suggests that college students have difficulty using the PCS in mathematics, science, and engineering settings (Montiel et al., 2008, 2009; Sayre & Wittman, 2007). When working with the PCS, students are often constrained to activity and ways of thinking constructed during experiences with the CCS. Sayre and Wittman (2007) illustrated how an engineering student relied on the CCS in situations the authors considered more suitable for the PCS, ultimately hindering that student's ability to solve problems within engineering contexts. Whereas Sayre and Wittman focused on a student's use of the PCS (or lack thereof) in engineering contexts, Montiel et al. (2008) characterized problematic relationships between calculus students' meanings of function, the CCS, and the PCS. For example, the students' reliance on the vertical line test, a procedure based on graphing functions in the CCS, became problematic (to the authors but not to the students) when the authors posed function tasks that the authors considered to involve the PCS. When given the formula and the PCS graph of $r = 2$ (e.g., visually a circle), some students applied the vertical line test to conclude that the graph does not represent a function. Other students converted the formula to its CCS form (e.g., $x^2 + y^2 = 4$) and then applied the vertical line test to reach the same conclusion. Collectively, we take the above findings to suggest that students do not construct a PCS that is distinct from (but related to) their ways of thinking about the CCS and related concepts.

The fact that some students develop ways of thinking constrained to the CCS (Montiel et al., 2008; Sayre & Wittman, 2007) despite holding intuitive notions of the PCS at an early age (see Piaget, Inhelder, & Szeminska, 1960) is understandable given the almost exclusive focus on the CCS in K-12 mathematics. After working nearly exclusively with one coordinate system to organize the plane, we would be surprised if students did not come to think of *the plane* and *the CCS* as essentially equivalent. However, the plane and the CCS are not equivalent, and it is important that students construct the PCS in ways connected to, but not constrained by, their ways of thinking about the CCS. In order to better understand how students might construct the PCS in such a way, we took a fundamentally different approach in this study as compared to prior researchers (Montiel et al., 2008; Sayre & Wittman, 2007). A limitation of previous researchers' approaches is that they did not examine student understanding *during* instruction that addressed students' construction of the PCS. The researchers instead utilized an approach and questions that we take to presume some understanding of the PCS (e.g., asking students if a polar curve or equation defines a function). We did not presume that students hold a working image of the PCS and thus explored students' activity during instructional activities designed to give us insights into their construction of the PCS. In the following section, we outline our instructional focus and informing theories.

4. Instructional design and covariational/quantitative reasoning

We drew primarily on theories of *covariational* and *quantitative reasoning* (Carlson, Jacobs, Coe, Larsen, &Hsu, 2002; Smith & Thompson, 2008; Thompson, 2011) to design the instructional sequence. We intended to promote an explicit focus on

Consider a boat at sea using sonar to determine the location of other objects.

1) If the sonar device detects an object that is 2 miles away, draw all locations on the screen that could correspond to the object.
2) If the sonar device detects an object that is 3.5 miles away, draw all locations on the screen that could correspond to the object.
3) What other measurement might you add to the device that will allow the crew to convey the exact locations of the objects from parts (a) and (b)? Illustrate multiple locations for an object and the corresponding values for this location.
4) Draw a coordinate system that conveys how far any object is from the boat and the measurement identified in part (c). Identify and label 3 points on your coordinate system.

Fig. 1. The sonar problem.

identifying measurable attributes of a situation (i.e., quantities) and reasoning about how these quantities change in tandem (i.e., covariational reasoning). For instance, we repeatedly asked the students to describe calculations, products (e.g., formulas and graphs), and claims in terms of whatever situation and quantities they were envisioning. Previous researchers have illustrated that maintaining a focus on quantities and their relationships is critical to supporting students' understandings of rate of change (Carlson et al., 2002), function (Oehrtman, Carlson, & Thompson, 2008), various function classes (Castillo-Garsow, 2012; Confrey & Smith, 1995; Ellis, 2007; Moore, 2014), the fundamental theorem of calculus (Thompson, 1994b), and graphing in the CCS (Lobato, Rhodehamel, & Hohensee, 2012). Hence, our general interest was in how students' construction of the PCS (including complexities encountered in this construction) might unfold during an instructional sequence informed by theories of quantitative and covariational reasoning.

An important tenet of quantitative and covariational reasoning is that quantities and relationships between quantities are cognitive constructions (Thompson, 2011). When we state in the previous paragraph, "whatever situation and quantities [the students] were envisioning," we do not mean that students always envision a situation consisting of quantities or a well-developed network of related quantities. In fact, students often do not envision situations in ways that entail a well-developed network of related quantities (Moore & Carlson, 2012; Thompson, 2011). For this reason, we designed the instructional activities to first focus on constructing and exploring the two quantities used to organize the PCS (e.g., a pairing of a directed length and a directed angle measure) before transitioning to graphing relationships within this organization. In what follows, we provide a description of the opening sequence. We direct the reader to Moore, Paoletti, and Musgrave (2013) for an extensive discussion of the graphing relationships sequence.

Our primary goal for the opening activity was that the students encounter a situation necessitating the organization of the plane using two quantities: a radial component that provides the measure of a directed length from a fixed point (the pole) and an angular component that provides the directed measure of an angle's openness (counterclockwise) from a fixed ray (the polar axis). In hopes of affording such a situation, we asked students to determine what information a surface radar system on a ship, with the ship defining a center point (and ignoring the curvature of the earth), might give the user in order to determine the exact location of another object at sea (Fig. 1). We stipulated that the system measures how far a detected object is from the ship and prompted the students to determine a second quantity whose measure would determine an exact location of the object. When the students identified an additional quantity, we asked them to construct a coordinate system based on the two quantities (Prompt 4 was not displayed to the students until they had identified the additional quantity).

Five weeks prior to the teaching experiment, the students experienced angle measure instruction occurring over three class periods. We anticipated that the students would draw on these experiences to determine a second quantity by which to quantify the positions of the objects described in Fig. 1. During the angle measure instruction, we intended that the students quantify angle measure as an equivalence class of arcs; we intended that students come to understand angle measure, regardless of unit, as based in measuring an arc in a unit proportional to a circle's circumference or radius (Moore, 2013). As Thompson (2008) described, "degree measure and radian measure are exactly the same type of thing-a measure of subtended arc" (p. 36). Measuring an angle's openness in degrees involves measuring a subtended arc in a unit 1/360 times as large as the circle's circumference that contains the arc. Measuring an angle's openness in radians involves measuring a subtended arc in a unit $1/(2\pi)$ times as large as the circle's circumference that contains the arc, with this unit being equivalent to the radius of that circle and a length $180/\pi$ times as large as the arc corresponding to one-degree on that circle. Thinking about angle measure as an equivalence class of arcs is important for students to conceive of an angle's openness in terms of a multiplicative structure compatible with what Thompson (1994a) described as a reflectively abstracted constant ratio. Without a multiplicative structure in mind, students understand angle measures merely as references to a position on a circle (e.g., $\pi/2$ is at the top of the circle), to particular geometric objects (e.g., a circle *is* 2π radians and a line *is* 180 degrees), or to a vague space between two rays (Moore, 2013; Thompson, 2008).[1]

The fact that the PCS is based on a quantity that can be thought of as equivalence classes of arcs raises an issue that does not occur with the CCS; the PCS is based on a dimensionless quantity. Yet, we hypothesized that a student who understands angle measure as an equivalence class of arcs can come to think about locating and understanding points in the PCS in ways

[1] We direct the reader to Moore (2013) for a detailed description of the angle measure approach (including its mathematical merits, cognitive merits, and entailed mental actions).

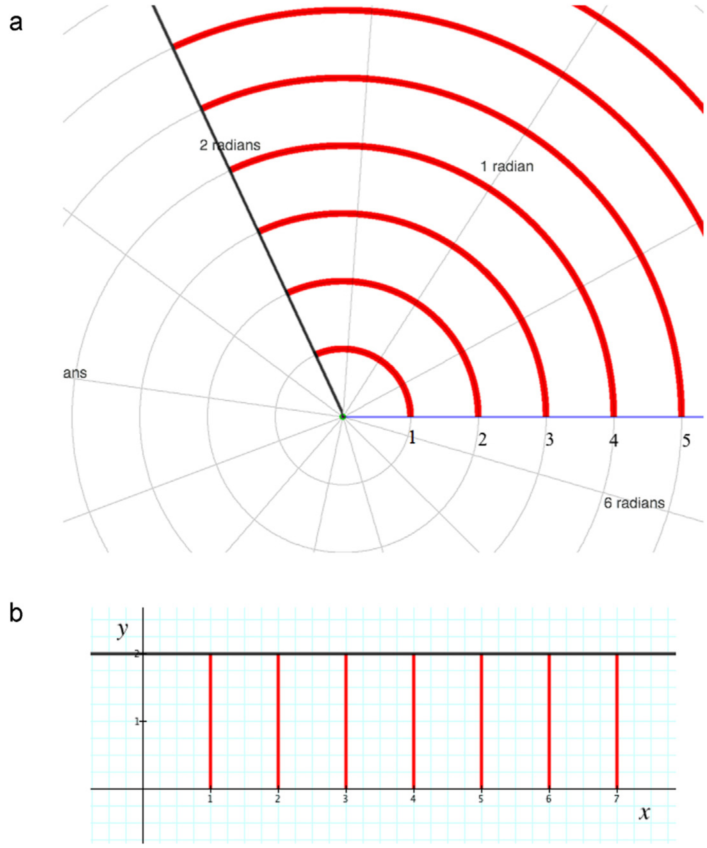

Fig. 2. Locating and understanding points in the (a) PCS and (b) CCS.

analogous to locating and understanding points in the CCS. For instance, consider the set of points defined by $(r, 2)$, $r > 0$. For each value of $r > 0$, the point $(r, 2)$ is at the terminus of a counterclockwise arc that: is on a circle of radius r units from the pole, starts at a directed length r units along the polar axis, and has a length of 2 when measured in radii (or $2r$ when measured in whatever unit r is measured). Hence, $(r, 2)$, $r > 0$ is the set of those terminus points on all circles of radius $r > 0$, with this set producing a ray that is orthogonal to each circle and subtending an arc of 2 radians – an arc length of 2 radii on the circle that contains that arc (Fig. 2a). This way of thinking about points in the PCS parallels understanding $(x, 2)$ in the CCS as defining the set of all points that are a directed length (analogous to the directed relative arc) of 2 units from the x-axis (analogous to the polar axis). For each value of x, the point $(x, 2)$ is at the endpoint of a directed segment that: is on a line x units from the y-axis, starts at a directed length x units along the x-axis, and has a length of 2 when measured in the same unit as the y-axis. Hence, $(x, 2)$ is the set of those endpoints, with this set producing a line orthogonal to the y-axis and 2 units from the x-axis (Fig. 2b).

Following the opening activity, we continued to explore the quantitative organization of the PCS by asking students to consider how changes in the values of a coordinate pair correspond to a point's position in the plane (Fig. 3). Our goal of this activity was not that students describe a point as 'moving around', 'moving out', 'moving in', or some other phrase denoting the physical motion of a point in the plane. We instead pushed students to describe changes in the point's location in terms of increases and decreases in the relevant quantities; we aimed to maintain that the students hold an image of a point as the result of projecting two quantities' measures (or magnitude). As an example, for an increase in n on (a), our goal was that the students reason that the radial value increases while holding in mind the angle measure – a relative arc that is measured

For the following coordinates in the polar plane, draw and explain the influence of increases and decreases in n.

a. $(r, \theta) = (3.5 + n, 2)$
b. $(r, \theta) = (3.5, 2 + n)$
c. $(r, \theta) = (3.5 + n, 2 + n)$

Fig. 3. Parameter problem presented to the students.

in a unit proportional to the radius (or circumference) of the circle containing that arc – remains constant, thus resulting in a point that is an increased directed length from the pole along the same ray as the original point.

After the above explorations, we concluded the instructional sequence by having the students covary radial values and angle measures in systematic ways so that they maintained and represented invariant relationships (e.g., $r = \sin(\theta)$ or $r = 2\theta + 1$). We also prompted students to relate this covariation to corresponding graphs in the CCS (e.g., reasoning that $r = \sin(\theta)$ and $y = \sin(x)$ convey the same covariational relationship but look different when represented in different coordinate systems). We emphasized a covariation approach in our work due to students often experiencing sustained difficulties reasoning about graphs as relationships between covarying quantities. These difficulties have negative consequences for students' success in STEM majors (Oehrtman et al., 2008). We also interpreted previous research in the area (Montiel et al., 2008) to imply that some students treat PCS graphs as objects independent of the (PCS) relationships these graphs represent. Thus, we intended that the students develop a fundamental understanding of graphs as the product of tracking covarying quantities within a particular coordinate system. Students who think of graphs as emergent relationships between covarying quantities are able to understand the PCS and CCS as different coordinate systems that can both be used to represent equivalent relationships (Moore, Paoletti, et al., 2013).

We note that a covariation approach differs from an approach that foregrounds analytically defining a curve in terms of both the CCS and PCS, and converting between these analytic definitions. The former approach foregrounds invariance at the level of covariational reasoning, whereas the latter approach foregrounds invariance at the level of perceptual shape (i.e., the curve) and rules for producing the shape (Moore, Paoletti, et al., 2013). We do not denounce the importance of defining a curve using multiple coordinate systems, but instead emphasize that we intended students develop an understanding of the PCS as a coordinate system used to represent relationships in ways that do not depend on characteristics of the CCS.

We provide the above analysis to clarify our approach to the PCS and to provide a backdrop for discussing the students' activities and experienced complexities reported here. Previous researchers (Montiel et al., 2008; Sayre & Wittman, 2007) investigated student thinking in the context of the PCS, yet these researchers were not clear about the ways of thinking that they considered to be entailed in understanding the PCS, nor did they describe the ways of thinking that they would like students to have for the PCS. It follows that these authors did not describe how such ways of thinking might develop or what complexities students might encounter in constructing these ways of thinking.

5. Subjects, setting, and methods

Our goal for this study was to better understand students' construction of the PCS while making no assumptions about their previous PCS experiences. To accomplish this goal we conducted a teaching experiment involving ongoing and retrospective analyses efforts for the purpose of building and testing models of student thinking (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Steffe & Thompson, 2000). The teaching experiment was situated in a mathematics education content course for pre-service secondary teachers at a large, public university in the southeast United States.

5.1. Subjects and setting

There were 21 (16 females, 5 males) students enrolled in the course. Each student was pursuing a Bachelor of Science in Education degree in secondary mathematics education (certification in grades 6–12). All students were third-year (in credits taken) undergraduates who had completed multi-variable calculus and at least two mathematics courses beyond the calculus sequence. The primary objective of the course was exploring reasoning processes critical for the teaching and learning of secondary mathematics. Theories of quantitative reasoning (Smith & Thompson, 2008; Thompson, 2011) and covariational reasoning (Carlson et al., 2002) informed the course design and goals. Prior to the teaching experiment sessions, the students engaged in activities designed to engender quantitative and covariational meanings for angle measure and trigonometric functions (see Moore, 2013, 2014).

During class sessions, the students worked primarily in groups of two to four. During this time, members of the research team circulated and facilitated the group discussions while trying to determine how the students were thinking about the tasks. The students shared 3-foot by 3-foot dry erase boards in order to capture their products, facilitate group discussions, and provide artifacts for consideration during whole class discussions. Whole class discussions followed group work and typically closed each class session. Whole class discussions involved the students presenting and analyzing each other's work while a teacher–researcher (the lead author) facilitated the conversations. During these discussions, the teacher–researcher often asked the students follow-up questions in order to probe and better understand the students' thinking, including their interpretation of each other's thinking. We hoped that doing so would provide us with deeper insights into students' thinking.

5.2. Data collection and analysis

The teaching experiment spanned five consecutive 75-min class sessions over the course of three weeks. During the teaching episodes, we sought to collect in-depth data on a few students as other studies have done (Montiel et al., 2008, 2009; Sayre & Wittman, 2007). We also sought to collect whole-class data to identify class-wide themes in students' ways of thinking. During whole class discussions, the observer–researchers videotaped all interactions and conversations. During group work, we focused data collection efforts (e.g., video and audio) on two student-pairs with the research team taking field notes on interactions with the other student groups. Each of the two student-pairs worked with a teacher–researcher while an observer–researcher videotaped their interactions to capture their writing, actions, and words. We chose the two student-pairs (Jack and Kate; Desmond and Penny) from a pool that volunteered for student-pair data collection. We chose and paired the students based on the results from a research-based, 25-item pre-assessment (Carlson, Oehrtman, & Engelke, 2010) that the students completed at the beginning of the course. We paired two higher scoring students (Kate and Jack scoring 16 and 20, respectively) and two lower scoring students (Desmond and Penny scoring 10 and 5, respectively).

The research team debriefed immediately after each teaching session in order to compare notes, to discuss and document possible models of student thinking, and to plan future instruction based on these hypothesized models. Our ongoing analyses also involved viewing teaching session videos between sessions and taking notes based on these viewings. Upon completion of the five sessions, research team members transcribed all videos in order to capture observable behaviors and audible speech. These transcriptions, in combination with the video files, field notes, and documented ongoing hypotheses of student thinking formed the data corpus for retrospective analysis.

Retrospective analysis efforts involved a combination of open and axial methods (Strauss & Corbin, 1998) and conceptual analysis (Thompson, 2000). We first analyzed each pair of students' words and actions in order to characterize their thinking. We then examined the whole group conversations to gain broader insights into the students' thinking. During our analysis efforts, which also involved comparing our characterizations to those developed during the ongoing phase of analysis, we identified common themes in our models of student thinking. Upon our identification of these themes, we searched the data for additional instances that supported or contradicted these themes. In the instances that contradicted our models of student thinking, we attempted to explain differences in the student thinking that generated each instance. Engaging in this iterative process, which is compatible with Thompson's (2000) notion of conceptual analysis, enabled us to build viable models of student thinking (e.g., models that explained the students' observable and audible actions) (Steffe & Thompson, 2000). For instance, as reported in Moore, Paoletti, et al. (2013), we characterized two students' repeated use of covariational reasoning when they graphed relationships in the PCS and compared their activity to graphing relationships in the CCS. Our focus in this article is on our analyses of three complexities that the students encountered during the teaching experiment.

6. Results

We describe three complexities students encountered while constructing the PCS: understanding radian angle measure, overcoming input–output conventions for functions in the CCS, and differentiating the PCS pole from the CCS origin. These complexities spontaneously emerged from students' activity; we did not intentionally design our tasks to raise these issues. Because of the spontaneous nature of these complexities, we consider them ideal exemplars for better understanding the process through which some students construct the PCS.

6.1. Radian angle measure and the PCS

A notable theme in the students' activity was the influence of their meanings for angle measure. Kate and Jack displayed no signs of difficulty with the angular component in the PCS (see Moore, Paoletti, et al., 2013). Desmond and Penny struggled significantly, especially when considering various arc lengths subtended by an angle of constant openness. We discuss Desmond and Penny's activity during the opening sonar task (Fig. 1) to illustrate the root of their difficulty. Prior to Excerpt 1, the pair had decided to use angle measure in radians as a second quantity for the radar system. The students then began to consider the process of determining the coordinate pair for a specific point in the plane (Excerpt 1).

Excerpt 1: *Penny and Desmond discuss how to label a point on a circle using angle measure*

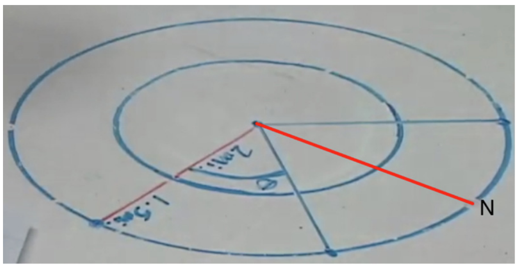

Fig. 4. Desmond and Penny's circles with radii 2 and 3.5 miles.

[*The students have on their board circles with radii measures scaled to represent 2 miles and 3.5 miles and what the teacher-researcher (T-R) understood as rays defining 1 radian and 2 radians – see* Fig. 4, *not to scale. They then attempt to identify the coordinates for a point 3.5 miles along a due North ray*.]

T-R:	What were you thinking there [*referring to her drawing the due north ray*], Penny?
Penny:	I was saying how many of our radians can we fit [*tracing along an arc of the 3.5-mile radius circle from the ray pointing due east to the ray pointing due north*].
Desmond:	That's what I was going to say. If, if you made it into-
Penny:	[*measuring the arc length from the ray pointing due east to the ray pointing due north on the 3.5-mile radius circle by iterating a piece of waxed string that they used as the radius of the 2-mile circle*] One, two, oh, is that, that's close to three [string lengths].
Desmond:	Yeah.
Penny:	So we know-
T-R:	So you're saying if we were, you're focusing if we were right here [*indicating the location where the 3.5-mile radius circle intersects the ray pointing due north*]?
Penny:	Yeah, well that's, I'm just saying that's our ninety degrees.
T-R:	Oh, okay. So ninety degrees.
Penny:	So if we have three six nine twelve [*pointing to locations on the 3.5-mile radius circle corresponding to due north, west, south, and east*], we have twelve, rad, radians, right?
[*After working on some calculations the conversation regarding radians continues*]	
T-R:	So how are you using "radian" right now? The word radian, how are you using that?
Desmond:	We're using radian in that the fact that this [*indicating with his fingers a length equal to the radius of the 3.5-mile circle*], we're making the three and a half miles into one unit.
T-R:	Okay-
Desmond:	And then we're, this, this is one [*motioning over the radius length of 3.5 miles*]. And then we're taking that one unit. Whatever, oh crap. We didn't, we used this [*the length of 2 miles*]-
Penny:	Two miles.
Desmond:	Instead of, instead of three and a half. Well you're right, this [*the 2 mile length*] would have worked for this [*referring to the 2-mile radius circle*] though, because say you would've went like this [*using a waxed string with a length of 2 miles to wrap around the arc of the 2-mile radius circle*], and then you would've kept going.
Penny:	What do you mean? No we didn't. Because this angle measure [*pointing to the angle labeled theta in* Fig. 4], whether it's here [*pointing to the arc intersected by the angle on the 2-mile circle*] or here [*pointing to the arc intersected by the angle on the 3.5-mile circle*], is the same angle. This is, our one radian is two miles [*showing the waxed string has a length of 2 miles*].
Desmond:	Okay. That's right. All right, you use this [*referring to the 2-mile radius length*] regardless. You're right, that makes sense. Okay.

We interpret Penny's actions to suggest that she conceived "one radian" as a fixed length of 2 miles, which was the radius of the first circle they constructed. When determining a coordinate pair corresponding to the intersection of the due North ray and the circle of radius 3.5 miles, Penny used a length equivalent to 2 miles (the radius of the first drawn circle) to conclude that a position due north (one-quarter of a rotation) on the 3.5-mile circle corresponds to approximately 3 radians. She then used this fact to conclude there were 12 radians for the entire 3.5-mile circle. Desmond's actions suggest that he considered using a unit length equivalent to the radius of whichever circle contained the arc they were considering, which is a foundational way of thinking for understanding radian measures as equivalence classes (Moore, 2013). But, by the end of this interaction, Penny's claim that "one radian" is fixed at a length of two miles swayed Desmond (e.g., "You're right, that makes sense").

Desmond and Penny created a paradox in their coordinate system by settling on a fixed unit length for measuring arcs and hence the openness of an angle, as opposed to choosing a unit length that varies proportionally with the radius of a circle. By choosing a fixed unit length, the students interpreted arcs on different circles subtended by the same angle as producing different angular coordinates. Fig. 5 represents an analogous situation to Penny and Desmond's system in which the radial coordinate unit length is also used as the unit length for measuring arcs regardless of the circle under consideration. For the angle shown in Fig. 5, within a system like the pair constructed the coordinate pair (1, 1) represents the point where the terminal ray intersects the circle with a radius of 1 unit, and the coordinate pair (2, 2) represents the point where the terminal ray intersects the circle with a radius of 2 units. This creates the problem of an angle with constant openness having both a measure of 1 radian and 2 radians (and, in fact, an infinite number of radian measures). Because of this, Penny and Desmond later questioned the use of radian measure (as they had conceived it) as the second quantity for conveying the location of a point in the plane.

As the teaching experiment progressed, we worked with the students to develop radian measure as defining an equivalence class of arcs. A description of our intervention with Penny and Desmond is beyond the scope of this article, but our work was consistent with that described in Moore (2013). We note that our progress with Penny and Desmond was neither trivial nor quick, with their difficulties understanding a quantity based on measuring in a unit length that varies depending on a circle's radius extending over several sessions. The students explicitly acknowledged this difficulty during the second teaching session when Penny stated, "See that's what confuses me, because how can we say that the length, I mean it's, I understand that [the arc length] is the same length as one radius, one radian [*referring to a subtended arc on a specific circle*]. But the angle measure is also one radian, so they're all [*referring to several subtended arcs*] one radian?" As the discussion and sessions continued, Desmond and Penny only came to understand radian measure as an equivalence class of arcs through

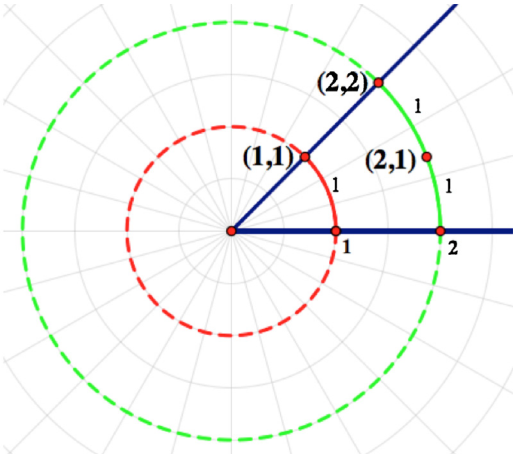

Fig. 5. A representation of using a fixed unit length to measure arcs.

repeatedly engaging in measuring arcs on different circles in a unit equivalent to the length of the radius of the circle containing the arc and reflecting on the outcome of this process (e.g., obtaining equivalent radii measures for arc lengths on different circles but subtended by the same angle).

6.2. Input–output conventions

Whereas we hypothesized that a sophisticated understanding of radian angle measure was a prerequisite for developing sophisticated PCS understandings, unexpected complexities arose when students attempted to think about the PCS in terms of their understanding of the CCS. The students' complexities involved coordinate ordering, variable use, and input–output conventions.

6.2.1. Coordinate ordering and input–output

In the United States, school curricula typically maintain the CCS conventions of designating the vertical axis as the output quantity and the horizontal axis as the input quantity, using the variables y and x to represent associated measures, respectively. Hence, coordinate pairs (x, y) have the form (input, output). This input–output dependency is reinforced by formulas written in the form 'y = *some expression in* x'. With respect to the PCS, the conventional coordinate ordering is (r, θ). Thus, considering functions such that θ is the independent or input variable creates the coordinate ordering (output, input). This departure from CCS conventions led to complexities for Desmond and Penny.

When asked to determine an analytic expression for $r(\theta)$ that produces the graph in Fig. 6 (e.g., $r(\theta) = 2\theta - 0.5$, $\theta \geq 0$), Desmond and Penny first determined the rate of change for the two quantities r and θ. To do so, they used the two-point formula that was, to them, tied to the CCS ordering of coordinates (Excerpt 2).

Excerpt 2: *Desmond and Penny use the slope formula* $(y_2 - y_1)/(x_2 - x_1)$ *to find rate of change*

Desmond:	Yeah, we can use, what, we can use *y-two* minus *y-one*. Find the slope. Cause it's the same as using the Cartesian.
Penny:	Sure, sure.
Desmond:	Okay.
Penny:	So, well, do like, um, change in, change in r over change in theta (*writing* $\Delta r/\Delta\theta$) or is it vice versa?
Desmond:	Well see, cause theta is in the y, r-theta [*referring to the ordering of the coordinate point*]. Is it, is it-
Penny:	Yeah, it's r-theta.
Desmond:	Okay, yeah. So changes in, changes in y over changes in x, is that what it is? [*Penny changes the formula to be* $\Delta\theta/\Delta r$ *representing a change in the second coordinate divided by the change in the first coordinate*]
Penny:	Yes.
Desmond:	Yeah, there we go. Awesome.

In this interaction, the students understood rate of change (or "slope", which itself implies the CCS) in terms of calculating a value using a formula tied to CCS coordinate ordering. To modify the formula for the PCS, the students replaced the y and x values in the formula $(y_2 - y_1)/(x_2 - x_1)$ with corresponding θ and r values, respectively (e.g., the second and first coordinates of the coordinate pairs, respectively). Doing so yielded the rate of change value of θ with respect to r (e.g., the change of θ is 0.5 times as large as the change of r), as opposed to the rate of change value of r with respect to θ (e.g., the change of r is 2 times as large as the change of θ), but we were unsure whether the students understood that this was the case.

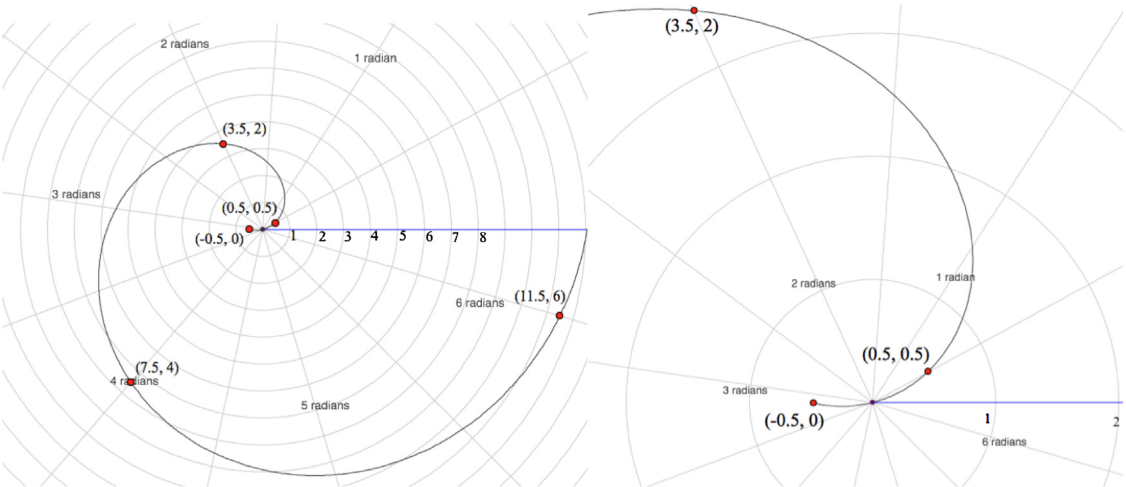

Fig. 6. The graph presented to the students.

Desmond attempted to move forward with the rate of change of 0.5 by determining a formula of the form $r(\theta) = 0.5\theta + b$, which we took to suggest that Desmond was not aware of what we understood as a discrepancy from their calculated rate of change and how they were using this rate of change. Penny, on the other hand, remained unsatisfied with their answer and she returned to the input and output dependency implied by $r(\theta)$ (Excerpt 3).

Excerpt 3: *Penny relying on input–output to find rate of change*

[*Interaction occurs immediately after Excerpt 2*]

Penny:	You said change of output over change in input. This is output [*writing "output" over the r in the pair* (r, θ)].
Desmond:	You're right.
Penny:	And this is input [*writing "input" over the* θ *in her polar ordered-pair* (r, θ)].
Desmond:	Yep.
Penny:	So it would be change in *r* over [*rewriting* $\Delta r/\Delta\theta$].

Whereas Penny and Desmond previously computed rate of change in a way that was tied to CCS conventions and coordinate pair ordering (Excerpt 2), Penny reframed the calculation in terms of input and output quantities (Excerpt 3). By conceiving of rate of change in terms of input and output quantities, as opposed to a particular coordinate ordering, Penny was not constrained to the CCS convention of (input, output) and rewrote their rate of change formula in terms of the dependency implied by $r(\theta)$. The pair concluded their activity by determining the formula $r(\theta) = 2\theta - 0.5$ (see Section 6.3).

6.2.2. Designating input–output variables

Given that Penny and Desmond encountered a complexity that stemmed from defining coordinate values in different input–output orderings, we decided to further explore the students' input–output notions with respect to the PCS and CCS. As mentioned above, a convention of school mathematics is denoting x as the input variable and y as the output variable. With respect to the PCS, functions are commonly defined with θ as the input variable and r as the output variable. But, just as one can arbitrarily choose coordinate ordering with respect to input–output, one can choose any variable to represent input and output values. For example, it is mathematically equivalent to define the function f as $f(x) = 3x$, $f(y) = 3y$, $f(r) = 3r$, or $f(\theta) = 3\theta$. It is only with the conventional ways of thinking attached to the variables x, y, r and θ that people might tend to view these function definitions as distinct; with conventional ways of thinking attached to the variables, the graphical representations are visually different in spite of the fact that the relationship represented between the input and output quantities is the same. Hence, after the students had graphed several relationships with θ as the input variable, they likened θ to the x variable of the CCS. Because of this, we asked the students to graph relationships in which we had defined r to be the input variable in order to investigate the students' capacity to consider either variable as the input quantity.

We first asked the students to graph the relationship defined by $\theta(r) = r^2$. Jack and Kate immediately rewrote the equation to imply r is dependent on θ (e.g., $r = \theta^{0.5}$). They then related their equation to the CCS equation of $y = \sqrt{x}$ (Excerpt 4).

Excerpt 4: *Kate and Jack attempting to graph $\theta = r^2$*

Jack:	So the Cartesian one is gonna be [*pause*] um like y equals square root of *x* right?
Kate:	Yea.
[*Kate draws a CCS and writes the equation $y = \sqrt{x}$*]	
Kate:	So we needed, how does [*pause*]-
Jack:	Or should we do it, not necessarily like a function, should it be positive and negative kinda thing [*motioning his marker as if drawing the relation $y = \pm\sqrt{x}$*].
T-R:	Whatever you guys think. Approach it however you want.
Kate:	So we need to do [*pause*], hold on, [*to Jack*] what was that function again?
Jack:	Huh?
Kate:	[*to T-R*] What was the function again?
T-R:	Theta equals *r*-squared
[*Both students write the equation $\theta = r^2$ on the whiteboard*]	
Kate:	Theta equals *r*-squared. So let's just plug in, I mean it's easier just to plug in *r*'s.
Jack:	Um but then [*pause*]. Yeah, we'd have to work back to get the theta.
Kate:	Okay, so if we plugged in radius length of one.
Jack:	So, yeah, then our theta is [*pause*] make a table [*laughing*].
[*Kate makes a table with r and theta as the two labels and uses r values of 1 through 4*]	
Kate:	*r* is one [*Jack agreeing throughout*], then we have theta is one, so then we just go one one [*graphing the point* (1,1) *in the PCS*].
Jack:	Okay.
Kate:	*r* is two, theta is four.
Jack:	Sure. Wait, ahh [*pause*] is that right? Why was I thinking [theta] was square root of two? *r* equals square root of theta. Yeah okay, sorry, you're right.

Throughout the interaction, Jack's actions of rewriting the given function rule and relating the obtained formula (e.g., $r = \theta^{0.5}$) to graphing in the CCS (e.g., $y = \sqrt{x}$) indicate he conceived of θ as the input variable despite our use of function notation $\theta(r)$ in the problem statement indicating that r is the input variable. Kate, on the other hand, understood either variable as the input variable (e.g., "it's easier just to plug in *r*'s"). As the discussion moved forward, this led the students to a conflict in how they were considering the situation. When Kate suggested using r values as input, Jack's commitment to considering θ as the input variable required that he "work back to get the theta" because he envisioned a relationship that entails θ as the input variable (e.g., $r = \theta^{0.5}$). Thus, while Kate easily transitioned to considering r as the input variable, this interaction indicates that Jack's meanings were partially dependent on θ as the input variable.

Like Jack, Desmond and Penny began the task by taking values of θ as input to determine corresponding output values of r. Unlike Jack, the pair did not rewrite the given formula in a way that maintained the relationship between r and θ (e.g., $r = \theta^{0.5}$). Instead, they determined values of r using the formula $r = \theta^2$ (recall, the given formula was $\theta = r^2$). Their repeated use of this formula occurred despite efforts of the teacher-researcher to draw the students' attention to the fact that they had reversed the variables r and θ. When the students noticed that their formula differed from the given formula, Desmond immediately sought to define the relationship so that θ was the input variable, explaining, "It's theta equals *r* squared, so wouldn't *r* be the square root of theta, right?" Desmond then rewrote $\theta = r^2$ as $r = \theta^{0.5}$, and we took his collective activity to imply that his thinking was partially constrained to considering θ as the input variable. In contrast, Penny remarked, "We just did the math backwards" (Excerpt 5).

Excerpt 5: *Desmond and Penny's conversation about polar conventions*

Penny:	If we did that backwards, erase it.
Desmond:	Yeah, let me see. Cause he switched the inputs and outputs on us.
Penny:	Okay.
Desmond:	So that means we have to figure out, we have to do a table, like-
Penny:	Uh okay, why don't we just do it backwards, why don't we do-
Desmond:	So *r* equals square root of theta (writes the equation $r = \theta^{0.5}$).
Penny:	Why don't we just do, put in for *r* and get theta.
Desmond:	Oh yeah, that's a, a perfect idea, that's fine, that works.

Like Jack (Excerpt 4), Desmond intended to maintain θ as the input variable by rewriting the equation to imply that r is in terms of θ. Conversely, and like Kate, Penny considered θ as dependent upon r as a viable way to think about the relationship. It was not until Penny made her suggestion to substitute values for r that Desmond considered r as a viable input variable, with his continued actions suggesting that he was more committed to considering θ as an input variable.

6.3. The CCS origin versus the PCS pole

Another example in which students' experiences with the CCS influenced their developing PCS understandings occurred as students considered whether a given graph contains a point corresponding to the PCS pole. Whereas the origin in the CCS is uniquely represented by the coordinate (0, 0), any coordinate of the form $(0, \theta)$ represents the PCS pole.[2]

[2] We note that a previous activity raised that any point in the PCS is represented by an infinite number of coordinate pairs (e.g., $(r, \theta \pm 2n\pi)$ for $n \in \mathbb{Z}$). However, we did not explore the pole during this part of the instructional sequence.

The pole (or "origin" as referred to by the students) created a complexity for students when we provided a graph (Fig. 6) and asked the students to determine a formula to define the graph (e.g., $r(\theta) = 2\theta - 0.5$, $\theta \geq 0$). After finding a constant rate of change for the relationship (see Excerpts 3 and 4), Desmond concluded that the graph passes through the "origin" and wrote $r(\theta) = 2\theta + 0$, basing his conclusion on the fact that a graph passing through the CCS origin includes the coordinate pair (0, 0). However, when the students attempted to verify their function definition using a given ordered pair, they changed their formula to $r(\theta) = 2\theta - 0.5$. This resulted in Desmond and Penny stating that the given graph (Fig. 6) was misleading in that it does not pass through the "origin" despite its appearance, and that instead the graph must pass through points very close to the origin.

Several groups reached a similar conclusion, and thus we decided to raise the issue of the "origin" during the subsequent whole class discussion. Taking a quick survey of the entire class, a majority of students concluded that the graph does not pass through the origin. Desmond was the first to make an argument, stating, "It doesn't go through the origin, because that wouldn't match our function [*referring to their final equation*]." When probed to justify his claim, Desmond argued, "If you use the origin zero-zero, then if, if your radius is zero and your theta is zero, there's no way you can get zero unless *b* [*referring to the b in* $y = mx + b$] is zero." A majority of the other students in the class then vocalized their agreement that the graph does not pass through the origin (Excerpt 6).

Excerpt 6: *Classroom discussion about whether the graph of the function* $r(\theta) = 2\theta - 0.5$ *includes the "origin" in the PCS*

Student A:	If you plug zero for theta, your *r* of theta will not be zero.
T-R:	Okay so you plug in zero for theta you're not going to get that [*referring to the pole*].
Student B:	Okay, for, it wouldn't be a function if you have two [*pause*] solutions, like two outcomes, when you enter zero. Like, the rad, the radians, if you enter theta as zero then you get two different radii [*referring to the points* (−0.5, 0) *and* (0, 0)] then it's not a function.
T-R:	So you're saying we'd have another point associated with [an input angle measure of] zero if it did pass through the origin.
Student B:	Yeah, and if this is, like we're graphing a function. It wouldn't be a function.
T-R:	Okay.
[*A third student makes an argument against the graph going through "the origin" based on the rate of change of the function*]	
Jack:	Can I say why it does go through the origin?
T-R:	You can say why it does, why you think it goes through "the origin".
Jack:	Any time the radius is equal to zero, the [*pause*] swirl goes through the origin, doesn't matter theta.
Kate:	Because there is no angle-
Jack:	There's no angle-
Kate:	if the radius is zero.
T-R:	What do you guys think about that?
[*Student's making noises to indicate that they agree with Jack and Kate's statements*]	
Desmond:	Oooohhhhh way to throw a wrinkle in it.
T-R:	So what do you guys think? So Jack, say that, say that a little louder, maybe, say that a little louder Jack.
Jack:	If at any point on *r*-theta, um-
T-R:	So you're saying if we have any point *r*-theta and we do what-
Jack:	Where *r* is equal to zero, it doesn't matter what theta is.

Like Desmond and Penny, a majority of the students conceived that a graph passing through the "origin" in the PCS required that the function include the pair (0, 0). Because (0, 0) was not a pair corresponding to their function, the students concluded that the given graph could not pass through the "origin" as they had conceived it. As part of their argument, the students used the property of a function that each input has a unique output to claim that it is impossible for the points (0, 0), and (-0.5, 0) to both be coordinate pairs associated with the function. The students' arguments are not incorrect from a function standpoint. Instead, the problem lied within their conception of the PCS "origin"; by thinking that (0, 0) is the only coordinate pair that corresponds to the pole in the PCS, the students viably concluded that the graph could not pass through the PCS pole until they understood other coordinate pairs as representing the PCS "origin".

In all, the classroom conversation regarding the pole not only led to the students revisiting their ways of thinking about function, it led to their comparing the underlying structure of the CCS and PCS, including relationships between the origin in the CCS and the pole in the PCS. Desmond later stated, "It's so hard to grasp that it goes through the origin but it's not zero-zero. I feel like that's ingrained in our mind," which we took to illustrate the importance of students considering the structural differences between the coordinate systems, and particularly their attending to the quantitative features of the CCS origin and PCS pole. To Desmond and the other students, the pole being all points $(0, \theta)$ was not an inherent feature of the PCS and instead a concept that they had to construct in a way that distinguished the PCS pole from the CCS origin.

7. Discussion

Our findings corroborate Montiel et al. (2008) and Sayre and Wittman's (2007) claims that more attention should be given to students' construction of the PCS. Our results extend their findings by identifying complexities that students encounter when constructing the PCS. For instance, our findings highlight how students' meanings for angle measure can influence their construction of the PCS. Additionally, the students' activities suggest that particular conventions of the CCS can become problematic when students construct the PCS. At the same time, each of these complexities provides students opportunities

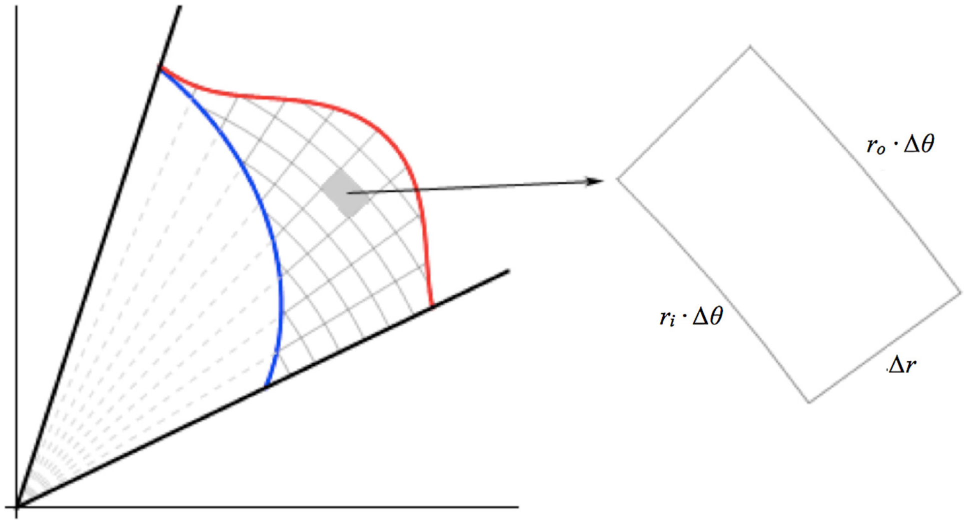

Fig. 7. Approximating area using the PCS (figure modified from Dawkins, 2014).

to reflect on their prior mathematical experiences in ways that further their understanding of angle measure, the CCS, and the PCS.

7.1. Giving attention to angle measure

A growing body of literature highlights the difficulties students have with angle measure, particularly that of coordinating radian measure with measuring arcs (Akkoc, 2008; Moore, 2013; Topçu, Kertil, Akkoc, Yilmaz, & Önder, 2006). Desmond and Penny's actions illustrate complexities that arise when students attempt to construct a coordinate system involving angle measures without foundational ways of thinking about angle measures and arcs. In the event that students have not conceived angle measures as equivalence classes of arcs, they can be left with paradoxes like that in Fig. 5 when constructing a coordinate system that involves arcs.

Given that students and teachers often hold fragile meanings for angle measure, like that of Desmond and Penny (Akkoc, 2008; Moore, 2013; Topçu et al., 2006), it leaves to question the nature of students' experiences with the PCS when their angle measure meanings are not explicitly addressed as they were in this study. For instance, when transitioning from integration of a region in the CCS to its counterpart in the PCS, students are expected to make sense of the PCS area element of $dA = (r \cdot d\theta) \cdot dr$, which differs from the CCS area element of $dA = dx \cdot dy$. With the PCS and radian measure tied to coordinating arc lengths and circles' radii, the area element can be thought of as representing the multiplicative area formed by the magnitude change in the radial direction of dr and the magnitude change in the angular direction of $r \cdot d\theta$ (see Fig. 7 for the non-infinitesimal approximation of the area element); the latter term emerges from the angular $d\theta$ representing a fractional amount of the radius and r representing the measure of the radius. Without an understanding of the PCS and radian measure that entails coordinating arcs and radii, students are primarily left with using expressions and formulas derived from the CCS and analytic techniques (via substitution or Jacobian determinants) to derive the corresponding PCS expressions and formulas. Or, as a more disheartening case reported by several of our participants when they reflected on their previous PCS experiences, students are left with memorizing sets of formulas and corresponding situations in which to apply these formulas.

We further inferred that students with robust angle measure meanings did not show an inclination to rely on the CCS when approaching the PCS tasks; these students did not impose the CCS on the plane during the tasks. Rather, they only drew on the CCS in order to compare activity in the CCS with their activity in the PCS (see Jack and Kate in Moore, Paoletti, et al., 2013). On the other hand, students with less robust angle measure meanings often relied on the CCS (with varying levels of success) when working on the tasks. Related to Sayre and Wittman's (2007) observation of a student being more inclined to use the CCS in situations that the authors considered more suitable for the PCS, we contend that a possible explanation for that student's actions is that the student held impoverished angle measure meanings. It is likely that, to a student with impoverished angle measure meanings, a coordinate system that entails using angle measures is not more suitable for describing situations quantitatively.

7.2. Conventions, coordinate systems, and learning opportunities

Coordinate systems have a rich history in mathematics, with both the CCS and PCS emerging during the 17th century (Eves, 1990). Since their inception, each coordinate system evolved with various conventions (that vary across fields), several of which we mention above. While practicing mathematicians and mathematics educators intend these conventions to primarily serve communicative purposes, as opposed to being absolute rules, our results indicate students develop ways of thinking in which conventions are inherent aspects of the mathematics; with respect to the students' ways of thinking, we argue that these aspects are not conventions but instead are necessary components of their assimilatory actions (Moore, Silverman, et al., 2013; Moore et al., 2014).

We are not surprised that what we consider to be conventions act as essential components of students' ways of thinking. Students' prior coursework and curricular materials likely maintained these conventions in all, or nearly all, cases. With respect to the CCS, one is hard pressed to find instruction, textbooks, or online narratives that do not maintain the input–output and variable conventions described here. When conventions are maintained in absolute, students are able to develop and maintain ways of thinking in which conventions and assumptions are implicit with little to no consequence, as their ways of thinking remain viable and useful. However, these ways of thinking do not support students in approaching situations that differ from the conventions and assumptions upon which those ways of thinking were constructed (Moore, Paoletti, et al., 2013; Moore, Silverman, et al., 2013; Zazkis, 2008). As Zazkis (2008) described, "there are constraints that are embedded within conventional understanding of situations" (p. 150). Or, as Moore, Paoletti, et al., (2013) argued when characterizing the phenomenon of students developing ways of thinking particular to a single coordinate system, "[students] are posited to solve problems set within the system in which those ways of thinking were developed. But when they move to a different system, ways of thinking that inherently involve activity dependent on system conventions become problematic" (p. 471). As one example of this phenomenon, Desmond had constructed a procedure for calculating the rate of change of a function so that the input–output dependency of this rate of change value was implicit in his way of thinking. Thus, he was constrained to the coordinate ordering of (*input*, *output*) (see Excerpt 2). In contrast, Penny's way of thinking for rate of change entailed conceiving the calculation in terms of explicit input and output quantities that were not constrained to specific positions within a coordinate pair, enabling her to accommodate the change in coordinate ordering (see Excerpt 3).

As the students reconciled various complexities that they encountered, they brought implicit assumptions and conventions about the CCS and PCS to the surface for discussion and consideration. In discussing the potential of using multiple coordinate systems to improve student understanding, Zazkis (2008) claimed,

> According to Skemp, "to understand something means to assimilate it into an appropriate schema" (p. 46). A question that pertains to mathematics teacher education is how can one understand better what has been already understood, that is, assimilated. I extend Skemp's claim by suggesting that to understand something better means to assimilate it in a richer or more abstract schema. I suggest that a richer schema is constructed when a mathematics concept in one's mind becomes a particular example of a more general mathematical concept. This happens, for example, when. . .familiar X-Y axes become an example of possible coordinate systems. (p. 154)

Zazkis continued by suggesting that supporting students in constructing a "richer schema" involves creating a perturbation that entails a perceived problem. We argue that the complexities presented above created such perturbations, and that the students reconciling these perturbations involved their constructing more robust understandings (or richer schema) of the underlying quantitative structures for the PCS and CCS (as well as angle measure), and thus coordinate systems as a whole.

Collectively, the students involved in the present study encountered complexities involving function, rate of change, and the structure of coordinate systems that led them to identify several so-called facts and procedures that they previously took true in absolute to actually be tied to CCS conventions. As an illustrative example, the students took a function passing through the pole in the PCS to include the point (0, 0) based on their experiences in the CCS. The complexity that resulted from this assumption (e.g., a graph's appearance contradicting this conclusion) generated an opportunity for the students to explore and reflect on the CCS origin and PCS pole. As another example of the interplay between the CCS and PCS that came from the same data set but was not presented above, students' experiences in the CCS led them to expect a "line" when graphing a relationship like $r(\theta) = 2\theta + 1$ (Moore, Paoletti, et al., 2013). The students reconciled their not obtaining a "line" in the PCS by concluding that graphing a linear relationship involves capturing a constant rate of change between covarying quantities, as opposed to producing a particular geometric object. Examples like these indicate that students can benefit from experiences with multiple coordinate systems because these experiences provide students with opportunities to develop understandings entailing quantitative structures and relationships that are not constrained to a single situation or representation and its conventions (e.g., understanding rate of change in terms of a relationship between inputs and outputs as opposed to a calculation tied to a particular coordinate ordering).

8. Final remarks

The theory of quantitative reasoning rests on the stance that quantities do not exist in some ontological reality, but are instead objects of the mind (Thompson, 2011). It follows from this perspective that coordinate systems, which involve the coordination of quantities, must be cognitively constructed, with this construction occurring over time and through students' repeated experiences with organizing the plane. We provide some insights into complexities that arise during this construction. Because this study was conducted with only a small group of students over a relatively short period of time, we have much to learn about students' construction of the PCS and coordinate systems in general. In this section we provide a few ideas for future research.

One limitation of the present study was that the students had previous experiences with the PCS, although these experiences did not seem to influence their activity during the teaching sessions. Because this study was with only one group of students who were in advanced stages of their undergraduate studies, future research should investigate students' initial construction of the PCS and their application of the PCS in STEM courses. For instance, one productive line of inquiry is to investigate students' ways of thinking about the PCS in the context of calculus ideas. In addition to providing insights into students' ways of thinking about the PCS, including the PCS's inclusion of angle measure as a coordinate quantity, such studies have the potential to provide additional insights into how students relate their PCS activities to their CCS activities in the context of calculus ideas.

Future research should investigate the potential of using multiple coordinate systems to provide students the opportunity to construct concepts by distinguishing what is essential to a concept and what is contingent on (what the student understands as) a representation of that concept. Our results suggest that exploring underlying differences between the CCS and PCS can simultaneously support students' understanding of both systems and further develop their understandings of related mathematical ideas (e.g. rate of change). In addition to corroborating other researchers' arguments that using multiple coordinate systems has instructive value by bringing tacit mathematical features and conventions to the surface (Moore, Paoletti, et al., 2013; Zazkis, 2008), this outcome resonates with research and perspectives on multiple representations (e.g., Dienes, 1960; Goldenberg, 1995). Specifically, our work aligns with Thompson's (2013) claim that students have an opportunity to develop a subjective sense of invariance when working in multiple representational contexts, with this subjective sense of invariance being the basis of their abstraction of mathematical ideas and concepts. Researchers who establish lines of inquiry in this area have the opportunity to explore learning that stems from students reflecting on their activity among coordinate systems, and particularly that of students reconciling complexities when constructing a new coordinate system.

The function concept provides one particular area that future researchers might focus on with respect to investigating student learning and the use of multiple coordinate systems. Our interest was not in exploring students' function concept per se, but some tasks involved functions and thus the students' activity was reliant on their meanings for function and related concepts (e.g., variables). In a separate study, we characterized Jack and Kate's (Moore, Paoletti, et al., 2013) reasoning about function and covariation. Those results combined with the present article suggest that future research should investigate the relationships between students' function concepts, their understanding of various coordinate systems, and the conventions that surround their activity when working in these coordinate systems. Function is an abstract notion and the use of multiple coordinate systems can potentially foreground various aspects of function (e.g., covariation, a mapping between two quantities' values, and variable use) in ways that aid students in constructing ways of thinking that are not dependent upon one particular representation or coordinate system. Researchers that explore the use of multiple coordinate systems in this area might produce critical insights into students' function construction, including how their meanings for function relate to their use of multiple coordinate systems.

Lastly, although we did not adopt a *transfer* perspective of knowledge in this article, the transfer perspective provides another way to conceptualize some of the students' activities. Most relevant to the present study is the actor-oriented perspective of transfer (Lobato, 2008), which distinguishes between an observer–researcher's perspective and a learner's perspective. As Lobato (2012) explained, an actor-oriented perspective of transfer enables researchers to approach knowledge (or knowing) as idiosyncratic to the knower and to characterize the "unexpected ways in which people generalize their learning experiences" (p. 236). The students in our study generalized their experiences with the CCS in several ways that were both unexpected to us and inconsistent with the meanings that we had hoped the students construct for the PCS. Yet, to the students (actors), these generalizations were sensible and internally consistent, and they remained so until the students (actors) experienced some form of cognitive disequilibrium that resulted from their conceiving a situation in which these generalizations were not appropriate. This outcome echoes Marton's (2006) claim that students' attempts to transfer knowledge can yield learning opportunities as the students come to understand similarities and differences between previously experienced and novel situations (e.g., the CCS and PCS, respectively). We envision that researchers who explore students' coordinate system understandings from a transfer perspective will not only contribute to the research base on students' understanding of coordinate systems, but also to our understanding of the interplay between transfer and learning.

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