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Chapter 7
Graphical Shape Thinking and Transfer

Kevin C. Moore

It is widely acknowledged that a learner's currently held cognitive structures afford and constrain her future learning experiences. It is also widely acknowledged that a learner's present learning experiences can shape and modify her previously constructed cognitive structures. Researchers refer to these phenomena in ways dependent on their theoretical framing. Researchers adopting a transfer perspective often appeal to processes of forward transfer and backward transfer to explain these phenomena (Hohensee, 2014; Lobato, 2012). Researchers adopting a Piagetian constructivism lens are disposed to explain these phenomena in terms of assimilation and accommodation (Piaget, 2001; Steffe & Olive, 2010; von Glasersfeld, 1995). Because each of these processes is influential in a learner's mathematical development, researchers have called for more detailed explanations of them in terms of specified mathematical content, concepts, and teaching (Diamond, 2018; diSessa & Wagner, 2005; Ellis, 2007; Hohensee, 2014; Lobato, Rhodehamel, & Hohensee, 2012; Nokes, 2009; Thompson, 2013b).

I address the aforementioned call in the present chapter by discussing forward and backward transfer in the context of students' meanings for graphs. I do so with three related goals. First, I define and elaborate on constructs, which are forms of what is referred to as *graphical shape thinking* (Moore & Thompson, 2015)—that Thompson and I introduced as epistemic subjects to capture students' meanings for graphs.[1] Epistemic subjects (Steffe & Norton, 2014; Thompson, 2013a) are conceptual models that specify categorical differences among students' in-the-moment

[1] Thompson and I initially used *shape thinking* as the stem phrase for the constructs (Moore & Thompson, 2015). We have since updated the stem phrase to *graphical shape thinking* to emphasize our focus on quantitative relationships and their graphs, as opposed to the study of geometric shapes.

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meanings. Second, and relatedly, I situate these meanings in terms of specific transfer processes and their implications for student activity in novel situations. Third, I use a synthesis of a student's actions to assert the role sequential processes of forward and backward transfer can play in students' graphical shape thinking.

I structure the chapter as follows to accomplish these goals. I first provide two vignettes in order to introduce the two graphical shape thinking constructs and motivate a focus on particular aspects of transfer. A concise discussion of the theoretical underpinnings central to this chapter follows the opening vignettes. I subsequently describe the two graphical shape thinking constructs and, using accompanying student data, illustrate them in terms of students' transfer processes. Generalizing from these cases, I introduce a way to frame concept construction in terms of theories of transfer and graphical shape thinking, and I provide a data example to illustrate the productive nature of such a framing. As part of this discussion, I provide suggestions for future research.

7.1 Two Vignettes

The following vignettes are from task-based clinical interviews (Ginsburg, 1997) that occurred as part of investigation into preservice secondary teachers' (PSTs') and undergraduate students' meanings for graphs in the context of noncanonical representations (Moore, Silverman, Paoletti, Liss, & Musgrave, 2019). PST1 is responding to the prompt and graph in Fig. 7.1a. PST2 is responding to the prompt and graph in Fig. 7.2a. Both vignettes are actual excerpts. As the reader engages with the vignettes, consider the particular meanings the PSTs are drawing on in that moment of reasoning, as well as the potential influence of previous learning experiences with respect to each PST's reasoning.

7.1.1 Vignette 1 (PST1): Where the Slopes Were[2]

Vignette 1 concerns the following prompt: *You are working with a student who happens to be graphing y=3x. He provides the following graph* [shown as (a) in Fig. 7.1]. *How might he be thinking about this?*

PST1: Um, like this [*rotating* Fig. 7.1a *90 degrees counterclockwise*—Fig. 7.1b—*laughing*].[3] Like I [*rotating back to* Fig. 7.1a], because if you turn it this way [*rotating to* Fig. 7.1b *again*], then this [*tracing left to right along hori-*

[2] "Int." stands for the interviewer.

[3] Throughout this chapter, as needed, I describe clarifications in participant explanations, gestures, and actions using "[*text*]". Italicized text indicates added information whereas standard text indicates our replacing an ambiguous word or phrase with my interpretation of her intended word or phrase. I also use this convention with a line break to indicate a summary of intermediate discussion.

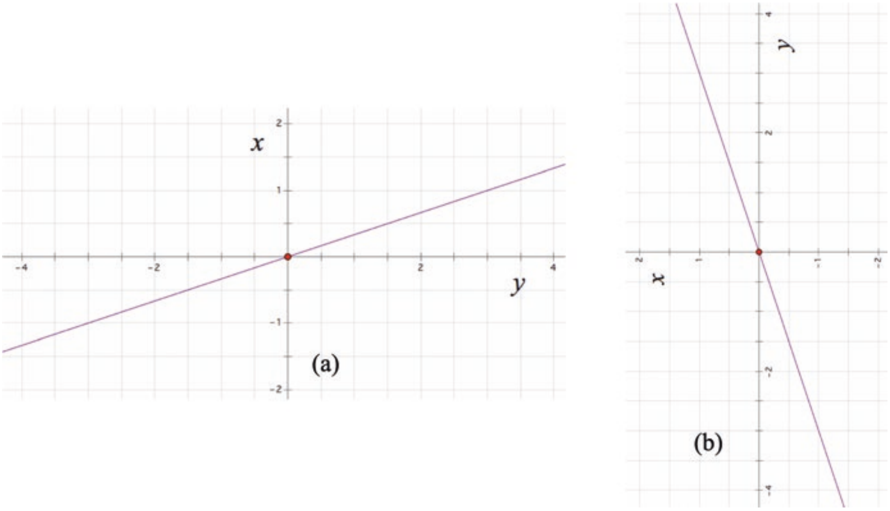


Fig. 7.1 (**a**) A hypothetical student's graph of $y = 3x$. (**b**) PST1's rotated graph

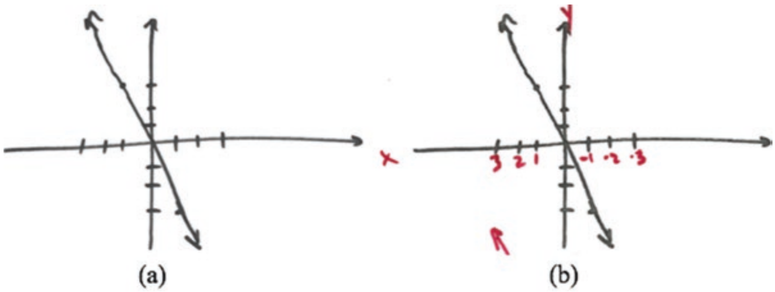


Fig. 7.2 (**a**) A second hypothetical student's graph of $y = 3x$. (**b**) PST2's added markings in red

zontal (x) axis] and then this [*tracing down vertical (y) axis*], it would be still not right though.

Int.: How would you respond to this student if they said, "Well, here's" [*rotating back to* Fig. 7.1a]

PST1: I mean I would tell them that they just labelled, like, well, I guess I would figure out what they were thinking about first because it could have just been something of they don't know which, they don't know that this is the *x*-axis [*pointing to the horizontal axis*] they don't know this is the *y*-axis [*pointing to the vertical axis*] . . . I don't really know if that makes sense. I mean the only way I can think of it is like this [*rotating to* Fig. 7.1b] and it's still wrong because this is negative slope [*laying a marker along the line sloping downward left to right*] . . . [*rotating back to* Fig. 7.1a]. I would just explain to them like the difference between the *x*- and *y*-axis . . . because if they were thinking of it as like sideways [*rotating to* Fig. 7.1b] or whatever [*rotating back to* Fig. 7.1a] it is, or inversely, or whatever, then, show them like the difference between like positive and negative slopes, also.

Because that's something that when I was in middle school we learned kind of like a trick to remember positive [*holds left hand pointing up and to the right*], negative [*holds left hand pointing down and to the right*], and no slope, and zero [*holding left hand horizontally*], like where, that's where the slopes were. And it's stuck with me 'til now, so it's important to know which direction they're going, when it's positive and negative and zero and no slope, too. But in this case positive or negative.

PST1 assimilated Fig. 7.1a essentially as a piece of wire with indexical associations of "slope" based on how it is placed in relation to two other pieces of crossing wires, as evidenced by her describing "where the slopes were," "which direction they're going," and using directional gestures to indicate a line's direction. PST1 understood rotating the paper as changing the line's slope (e.g., "which direction they're going") and she did not perceive any rotation to result in an image of a line associated with $y = 3x$. Furthermore, PST1 explicitly appealed to the previous learning experiences in which she formed these associations, thus anticipating the given task as resolved by producing a line in the "direction" learned during those experiences.

7.1.2 Vignette 2 (PST2): A Product of How We've Decided to Represent Things

Vignette 2 concerns the following prompt: *You are working with a student who happens to be graphing y=3x. He provides the following graph* [shown as (a) in Fig. 7.2]. *How might he be thinking about this?*

[PST2 has labeled the axes as shown in Fig. 7.2b; claims that the graph is of y = 3x].

Int.: So, what about a student who says, that says, "That can't be, that can't be right [*pointing at* Fig. 7.2b] because that's sloping downwards left to right. You know, that's going down to the right, so it can't be right. It has to be negative."

PST2: No, um, that sloping downward to the right [*moves hand down and to the right*] is a product of the convention of us labeling our axes with our positives over here [*motioning to her right*] and our negatives over here [*motioning to her left*], so you can look at it and we can trust that [*making hand motion down and to the right*] that's going to be a negative slope as long as everything is within our conventions.

Um, but slope is really just rate of change. And so, what this is telling us, this three [*circles* 3 *in equation*], is that when x, it's like [*writes "rise/run"*]. Is it sad that I still have to use rise over run like this? I feel like this is so bad [*writes* 3/1]. Well, anyways. Okay. So when we're saying that

> when our x, or our y changes on this graph, when our y changes by 3 [*pointing to the* 3 *in* 3/1], our x is changing by one [*pointing to the* 1 *in* 3/1]. So, if we can go up three [*pointing to the dash indicating a value of* $y = 3$ *on the y-axis*] in the positives [*puts plus signs beside* 3 *and* 1], we're still going positive one. But now our positives are over here [*motioning to her left*], so we have to be cognizant of the way our axes were labeled.
> If we were to switch this [*using her hands to indicate changing the orientation of the horizontal values*], it would flip and have that picture or image [*making hand motion up and to the right*] that you're looking for. But that's, again, just a product of how we've decided to represent things.[4]

PST2 assimilated Fig. 7.2a to a system of meanings based in images of coordinating quantities' values as they varied within an unconventional reference system as evidenced by her explicit attention to quantities' magnitudes and values both in her discussion and gestures. After this interaction, PST2 also sketched a graph like Fig. 7.1a and claimed it to be an alternative representation of $y = 3x$. Furthermore, PST2 explicitly raised issues of convention, suggesting that her previous learning experiences directed her attention to arbitrary representational choices when considering the viability of a novel solution. This enabled her to understand each graph (e.g., Fig. 7.1a and the conventional displayed graph for $y = 3x$) in terms of an equivalent relationship between covarying quantities, with perceptual differences between them resulting from different coordinate system conventions.

PST1's actions, which suggest establishing relations based in perceptual cues and figurative properties of shape, are consistent with what Thompson and I (Moore, 2016; Moore & Thompson, 2015) term *static graphical shape thinking*. PST2's actions, which suggest her establishing relations based in covarying quantities and how they are represented within a coordinate system's conventions, are consistent with what Thompson and I term *emergent graphical shape thinking*. The marked differences between the PSTs' meanings and established relations with their previous learning experiences raise several broader questions. Two questions are:

1. In what ways do students' graphical shape thinking influence their construction of relations of similarity between previous and current learning experiences (i.e., transfer)?
2. Relatedly, in what ways do students' attempted construction of relations of similarity between previous and current learning experiences (i.e., transfer) influence their development of graphical shape thinking?

[4] PST2 subsequently discussed "rise over run" as a convention itself and how the graphs are such that the variation in x relative to variation in y is 1/3 and the variation in y relative to variation in x is 3.

7.2 Theoretical Framing—Transfer, Understanding, and Meaning

The two questions raised in the previous section center on transfer, understanding, and meaning. Defined generally, transfer is "the influence of a learner's prior activities on his or her activity in a novel situation" (Lobato, 2008, p. 169). Educational research, particularly in mathematics education, entails numerous perspectives on transfer. Early researchers characterized transfer in ways that reflected an implicit or explicit assumption of there being objectively correct solutions to mathematical problems. Cox (1997) and Lobato (2006) identified that these early approaches to transfer had roots in associationism and behaviorism that can be traced to Thorndike's (1903, 1906) notion of *identical elements*. More recently, researchers have claimed to problematize the relationships between an external environment and the mind (see Anderson, Reder, & Simon, 2000), but Lobato (2006, 2012) and Wagner (2010) argued that these accounts do not problematize these relationships in practice, instead operating "as if situational structure could be directly perceived in the world" (Wagner, 2010, p. 447).

To be more sensitive to nonnormative reasoning or what an expert might deem "incorrect" reasoning, Lobato (2006, 2012; Lobato & Siebert, 2002) introduced the *actor-oriented transfer* (AOT) perspective. The AOT perspective explores transfer as perceived by the learner. It emphasizes a learner's construction of personal relations of similarity between learning experiences and, accordingly, clarifies that claims about the nonnormative (or normative) performances resulting from transfer are from the perspective of the observer; all activity is viable and normative from the perspective of the learner. The AOT perspective also frames transfer in terms of the construction and reconstruction of knowledge. Whereas traditional perspectives have approached transfer as a static application of knowledge, the AOT perspective approaches transfer in terms of active, subjective constructions of similarity. The AOT perspective thus accounts for forms of transfer that promote learning through cognitive reorganization and accommodation (Hohensee, 2014; Lobato, 2012; Lobato & Siebert, 2002). Reflecting this affordance of the AOT perspective, researchers' adoptions of the AOT perspective have yielded explanations of students' (and teachers') meanings and learning in numerous content areas (Diamond, 2018; Ellis, 2007; Hohensee, 2014; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002). Researchers have also used the AOT perspective to identify how particular artifacts, language, and other factors of classroom instruction can explain differences in students' transfer of learning (Lobato et al., 2012).

Because the AOT perspective seeks to explain transfer from the perspective of the learner, it is productive for a researcher to pair the AOT perspective with a framing of meaning that emphasizes its subjective nature; an appropriate framing of students' meanings provides the ground by which a researcher can situate accounts of transfer. In this chapter, I draw on Thompson and Harel's descriptions of meaning and understanding (see Thompson, Carlson, Byerley, & Hatfield, 2014). The distinction between understanding and meaning is rooted in Piaget's characterization of understanding as assimilation to a scheme and of meaning as the space of implications created by a moment of assimilation (Skemp, 1962, 1971; Thompson, 2013b; Thompson & Saldanha, 2003). Thompson and Harel defined *understanding* to refer to a cognitive, in-the-moment state of equilibrium that results from assimilation. Understanding could occur from having assimilated an experience to a stable scheme, or from a functional accommodation specific to that moment and arrived at by an effortful coordination of existing schemes (Steffe, 1991). For instance, a student could perceive two marks on a piece of paper as orthogonal and assimilate the marks as coordinate axes, thereby establishing a state of equilibrium (i.e., an understanding). If the student also perceives an unfamiliar curve within the assimilated coordinate system, he might engage in effortful activity to understand the unfamiliar curve. The student could attempt to relate the unfamiliar curve with the collection of shapes and associated perceptual properties with which he is already familiar through prior learning experiences. Or, the student could attempt to imagine the curve in terms of an emergent trace of covarying values within the respective coordinate system and relate that to previously experienced covariational relationships. Either could result in a state of understanding via assimilating the curve to a meaning.

Meaning in Thompson and Harel's system refers to the space of implications that a moment of understanding brings forth (Thompson et al., 2014). When a person creates an understanding by assimilating an experience (e.g., a perceived word, phrase, diagram, or set of statements) to a scheme, the scheme is that person's meaning in that moment; the person's meaning in that moment consists of an organization of actions, operations, images, and schemes that the person anticipates or enacts (Piaget & Garcia, 1991; Thompson, 2013b; Thompson et al., 2014). Establishing a state of understanding through assimilation to a meaning can occur in many forms. It can be a nearly subconscious, habitual act (e.g., reciting learned multiplication facts), or it can be an effortful progression of reciprocal acts of accommodation and assimilation (e.g., sustained problem solving).

Returning to the notion of transfer, and as I illustrate in this chapter, transfer can occur within either case of establishing a state of understanding through assimilation to a meaning. A researcher can identify different forms of transfer in order to characterize the influence and interplay of a student's meanings constructed during previous learning experiences and their current actions and learning experience. *Forward transfer* and *backward transfer* have emerged as two forms of transfer useful for characterizing such experiences. Hohensee (2014) introduced forward transfer and backward transfer to differentiate between the influence of a learner's prior conceptions and actions on her activity in a novel situation (i.e., forward transfer) and "the influence…new knowledge has on one's ways of reasoning about related mathematical concepts that one has encountered previously" (p. 136; i.e., backward transfer). Stated in terms of meanings, forward transfer is how previously constructed meanings influence the assimilation of a present experience. Backward transfer is how a novel experience and associated meaning influences the learner's previously constructed meanings.

With the constructs of forward and backward transfer formally introduced, I return to the opening vignettes. Recall that PST1's actions suggest her drawing relations of similarity based on previously learned associations between the perceptual direction of a line and "slope." This is a form of forward transfer. She experienced a novel axes orientation in the form of hypothetical student work, and her prior conceptions of "slope" entailed an axes orientation that required that she attempt to modify the graph so that they were relevant. Recall that PST2's actions suggest her drawing relations of similarity based on coordinating quantities' covariation with attention to coordinate conventions. This is also a form of forward transfer, but there are aspects of her actions that suggest backward transfer. Namely, PST2 called attention to the previously learned mnemonic phrase and calculation of "rise over run" being problematic in the context of the unconventional axes orientation. An explanation for her actions is that experiences with graphing in unconventional axes orientations influenced her meaning for the mnemonic phrase and calculation—one typically taught in Grades 6–12 curricula only in the context of conventional axes orientations—so that she came to understand it as subordinate to the concept of forming a multiplicative comparison. Her experiences with graphing covariational relationships in unconventional axes orientations entailed backward transfer with respect to her meaning for "rise over run" so that it could accommodate unconventional axes orientations, and the phrase no longer was absolutely literal relative to the implied physical movements.

As suggested by this interpretation of PST2's actions, transfer can involve accommodations to previously constructed meanings (Hohensee, 2014; Lobato, 2012). Transfer and accommodation can occur in the context of two (or more) concepts or topics. For example, Hohensee (2014) illustrated backward transfer in terms of how students' learning of quadratic relationships can influence their previously constructed meanings for linear relationships. Or, transfer and accommodation can occur in the context of the same concept experienced across many learning experiences, as illustrated by Lobato and Siebert (2002) and Lobato et al. (2012). Because graphical shape thinking primarily refers to meanings for the same concept (e.g., graphing), and after elaborating on each of the graphical shape thinking constructs, I highlight how forward and backward transfer can potentially relate to the development of students' graphical shape thinking.

7.3 Graphical Shape Thinking

Each form of graphical shape thinking represents an epistemic subject that stabilized across a research program initiated by Thompson (1994a, 1994b) and was then extended by Thompson, and other colleagues, and myself. Collectively, we targeted secondary students', undergraduate students', and teachers' meanings for precalculus and calculus ideas, including graphing (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, 2014, 2016; Moore, Paoletti, & Musgrave, 2013; Moore & Silverman, 2015; Moore, Silverman, Paoletti, & LaForest, 2014; Paoletti

& Moore, 2017; Saldanha & Thompson, 1998; Thompson, 2013b, 2016; Thompson & Carlson, 2017; Thompson, Hatfield, Yoon, Joshua, & Byerley, 2017; Thompson & Silverman, 2007). An epistemic subject is a characteristic of thinking that has stabilized within a researcher's thinking across the second-order models she has created for particular students' mathematical meanings (Steffe & Norton, 2014; Steffe & Thompson, 2000; Steffe, von Glasersfeld, Richards, & Cobb, 1983; Thompson, 2013a). An epistemic subject is a hypothetical way of thinking that proves increasingly viable through a researcher's use in predicting and explaining students' behaviors; it supports a researcher engaging in forward transfer as a mechanism to organize their experiences with future students. The generality of epistemic subjects empowers researchers and educators to interact with students in more productive and targeted ways (Hackenberg, 2014; Thompson, 2000).

Consistent with the AOT perspective, epistemic subjects are nonjudgmental with respect to what an observer might perceive to be correct mathematics; students' meanings are always considered viable from their point of view. Furthermore, characterizing a student's actions as consistent with an epistemic subject is not a statement about the student's capabilities or other potential meanings they hold and transfer. Students can, and do, hold multiple meanings for a concept, each of which are products of students having reasoned about that concept in particular ways. Any claim above regarding PST1, PST2, or an individual below is not a holistic claim about the individual but rather a claim about that individual's actions in that moment.

7.3.1 Static Graphical Shape Thinking

Static graphical shape thinking characterizes actions that involve conceiving a graph as if it is essentially a malleable piece of wire (*graph-as-wire*). Thompson and I (Moore & Thompson, 2015) chose the term *static* to indicate that a student assimilates a displayed graph so that he predicates his actions on perceptual cues and figurative properties of shape, and imagined transformations are with respect to physically manipulating that shape as if it were a wire (e.g., translating, rotating, or bending). Because static graphical shape thinking entails actions based in perceptual cues and figurative properties of shape, an element of thinking statically is that conceived associations are (in that moment of understanding) indexical properties or learned facts of the *shape qua shape*; the associations require further contextualization to entail logico-mathematical operations (cf., emergent graphical shape thinking in which quantitative operations constitute the meaning). To illustrate, PST1's treatment of "slope" suggests that her graph's defining properties were its straightness and its direction, and her graph's direction was associated with learned facts of slope. Her subsequent actions were to rotate the graph-as-wire, inferring that changing the line's direction changed its slope.

In addition to associations like slope, the facts of shape constituting static graphical shape thinking can be equations, names, or analytic rules. For instance, as reported by Zaslavsky, Sela, and Leron (2002), a student could associate a graph that he understands as a line with the analytic form $f(x) = mx + b$ regardless of coordinate system or axes' scales. Another student could assimilate a displayed graph as "curving up" and associate this with "exponential" and the analytic form $f(x) = a \bullet b^x$. In each case, the students' graphs entail indexical associations between shapes (e.g., "line" or "curve up"), function class terminology (e.g., "linear" or "exponential"), and analytic rules (e.g., $f(x) = mx + b$ and $f(x) = a \bullet b^x$); in the moment of assimilation, names and associated analytic rules are little more than memorized facts associated with various graphs-as-wire.

Because of its basis in perceptual cues and figurative properties of shape, static graphical shape thinking enables a learner to establish relations between learning experiences via foregrounding perceptual and figurative aspects of a graph. To illustrate, I first return to Lobato et al.'s (2012) findings. Recall that they identified how particular artifacts, language, and other factors of classroom instruction can explain differences in students' reasoning and transfer. Specifically, the authors likened some students' reasoning about a graph to reasoning about a "piece of spaghetti" (Lobato et al., 2012, p. 452) with a property of visual steepness. They classified such reasoning as focused on physical objects (as opposed to mathematical objects, as described in the next section). True to the AOT perspective, Lobato et al. (2012) illustrated that such reasoning did support students' transfer but that such transfer processes were for the purpose of describing properties of the "piece of spaghetti" (Lobato et al., 2012, p. 452). These students did count squares or boxes on coordinate grids to characterize slope, but they did so ignoring axes labels or scale. Such actions are a hallmark of static graphical shape thinking. Even in cases in which students do identify and reason about numbers, often to handle a perturbation in a novel task, they do so for the purpose of describing perceptual or figurative properties (i.e., how one moves or visual steepness).

As an additional illustration of static graphical shape thinking and forward transfer separate from slope, consider Excerpt 1 and Excerpt 2, which occurred during clinical interviews used to investigate PSTs' meanings for noncanonical displayed graphs (see Moore & Silverman, 2015; Moore et al., 2014; Paoletti, Stevens, Hobson, Moore, & LaForest, 2018). The interview prompt (Fig. 7.3) depicts hypothetical students claiming a Cartesian graph displays the sine function and its inverse simultaneously. We designed the hypothetical students' claim to reflect the understanding that the displayed graph is $[(x, y) \mid -\pi/2 \leq x \leq \pi/2, y = \sin(x), x = \arcsin(y)]$.

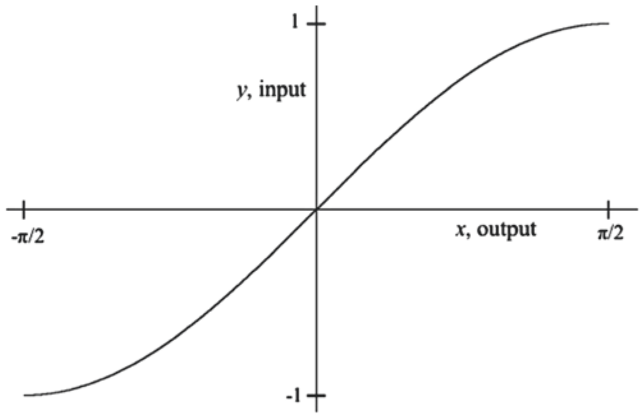
Excerpt 1: Brienne's response to the hypothetical students' statement that Fig. 7.3 represents the inverse sine function.

Brienne: I'm thinking this just kind of looks like the sine graph, like the plain sine graph [*laughs*]. Which is going to be different. So, no…

Excerpt 2: Sansa's response to the hypothetical students' statement that Fig. 7.3 represents the inverse sine function.

Sansa: It looks the same . . . the sine graph . . . I mean he graphed the sine graph . . . um, at pi over 2 sine is 1.

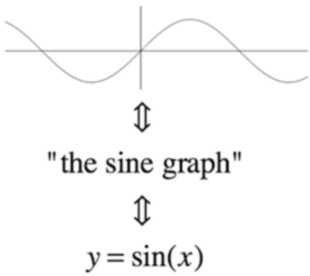
[*The researcher focuses Sansa on the students' statement about and labeling of input and output. She continues by rejecting the student's statement.*]



[The students] claim: "Well, because we are graphing the inverse of the sine function, we just think about x as the output and y as the input." When giving this explanation, they add the following labels to their graph.

Fig. 7.3 Graph and prompt posed to the students

Fig. 7.4 The sine shape qua shape



Sansa: You can't just label it like that. Um, why? Why can't you do that? I don't know. I feel like he's missing the whole concept of a graph . . . Like a sine graph's like a, it's a graph like everyone knows about, you know . . . that's just no. I think they're just missing the concept of graphing [*she continues to reiterate that the student graphed the sine graph and not the arc-sine graph*].

Both Brienne and Sansa's actions indicate their previous learning experiences having resulted in them associating a shape with a name or function (e.g., "sine" or "the sine graph"; Fig. 7.4). For instance, Sansa described her graph as follows: "looks the same . . . the sine graph . . . everyone knows about." I understood her to mean she perceived a learned shape that everyone including mathematics students should recognize as "the sine graph." The students' actions also suggest they had come to associate the name or function uniquely with the recognized shape; the shape could not be given a second name or function, and a different name or function should yield a different graph. This influenced how they perceived the viability of the students' claim, ultimately rejecting the given graphs as potentially representing the inverse sine function. For example, Brienne subsequently added that the "graph of an inverse function" should look different than the graph of the parent function.

7.3.2 *Emergent Graphical Shape Thinking*

Emergent graphical shape thinking characterizes a student's actions that involve conceiving a graph (either perceived or anticipated) simultaneously in terms of what is made (a trace entailing corresponding values) and how it is made (a sustained image of quantities having covaried). Thompson and I (Moore & Thompson, 2015) chose the term *emergent* to indicate that a student assimilates a graph—whether given, recalled, or constructed in the moment—as a trace in progress that is born or derived from images and coordination of covarying quantities. The student conceives the result of this trace to be the emergent correspondence between covarying quantities (Carlson et al., 2002; Frank, 2017; Saldanha & Thompson, 1998; Thompson et al., 2017). I illustrate states consistent with this meaning by showing

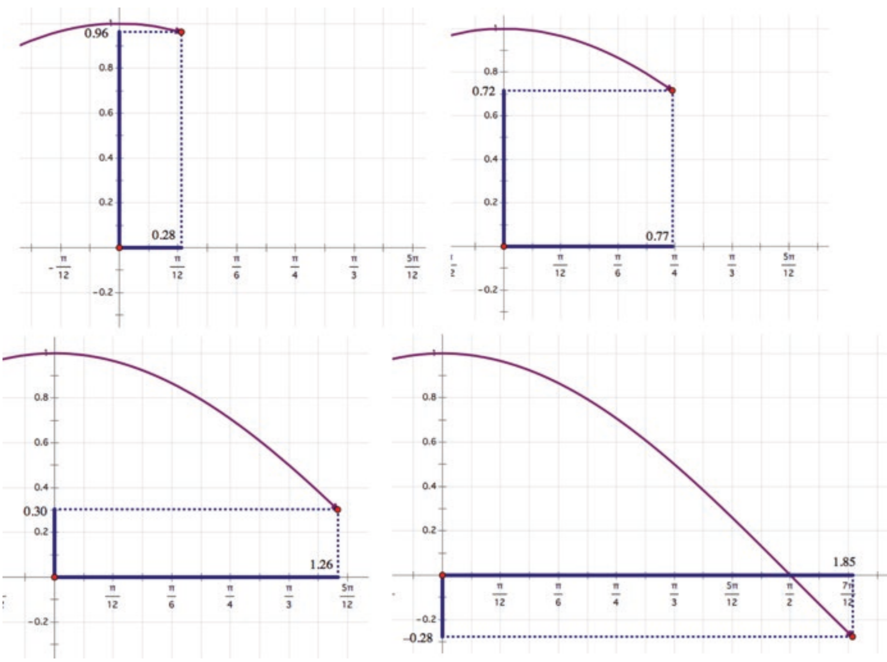


Fig. 7.5 Instantiations of emergent shape thinking

instantiations of an emergent trace of two quantities' magnitudes (Fig. 7.5), but note that static images alone are insufficient to convey emergent shape thinking. Emergent shape thinking is more complex than the displayed instantiations because it entails images of covariation: imagining magnitudes in flow, reasoning about what happens immediately after an instantiation, and reasoning about what happens between instantiations).[5]

An element of thinking emergently is that conceived features or attributes are properties of the covariational and quantitative operations used in assimilation; quantities and their covariation are organic to a student's graph when thinking emergently. Returning to Vignette 2, PST2's treatment of "slope" or rate of change suggests that her graph's defining properties were the covariation that produced it under the constraints of how the quantities were represented within particular axes organizations. Hence, PST2 understood traces in perceptually different orientations as representing equivalent properties of covariation (e.g., no matter the orientation, she conceived a displayed graph such that the rate of change between the two emergent quantities is three).

In addition to mathematical concepts like slope or rate of change, and because of its bases in schemes of covariation, emergent shape thinking associates function class terminology and analytic rules with images of covarying quantities (and the produced correspondence of values). Consistent with PST2, a student thinking emergently could understand a graph as representing a linear relationship of the form $y = mx + b$ via constituting a curve in terms of two values, y and x, covarying at a constant rate with a measure of m (or $1/m$ for changes of x measured relative to changes of y). As another example, a student thinking emergently understands a graph to be "exponential" and of the form $y = a \bullet b^x$ via conceiving a trace such that the rate of change of y with respect to x is proportional to y (Castillo-Garsow, 2010). In the moment of assimilation, images and properties of covariation form the basis of students' associations between their graphs, function class terminology, and analytic rules.

Because of its basis in quantitative and covariational operations, emergent graphical shape thinking enables a learner to establish relations between learning experiences via foregrounding the covariational properties that produce a graph. These properties are consistent with what Lobato et al. (2012) called mathematical objects. As an alternative to foregrounding a graph's "look" as in the case of Sansa and Brienne, consider Shae's focus on covariational properties when responding to the same prompt. Shae was a PST involved in the same series of studies as Sansa and Brienne (Moore, Silverman, et al., 2019; Paoletti, Stevens, Hobson, Moore, & LaForest, 2015). Shae first explained that if x represents angle measure values (in radians) and y represents directed vertical distance measures (in radii), then the sine function denotes x as an input value and y as output value, and the arcsine function

[5] I direct the reader to other work (Carlson et al., 2002; Castillo-Garsow, Johnson, & Moore, 2013; Confrey & Smith, 1995; Ellis, Özgür, Kulow, Williams, & Amidon, 2015; Johnson, 2012, 2015; Saldanha & Thompson, 1998; Thompson, 1994a; Thompson & Carlson, 2017) for more extensive treatments of the schemes and operations involved in covariational reasoning.

reverses these roles. She further explained that either axis could represent input or output values and therefore understood the graph as being both the sine and arcsine functions. At this point, I was not sure whether Shae conceived her displayed graph covariationally and I presented a canonical Cartesian graph of the inverse sine function (Fig. 7.6). I explained that a second student claimed it to be the graph of the inverse sine function, as opposed to the graph in Fig. 7.3. Shae understood both graphs to represent "the same thing" (Excerpt 3).

Excerpt 3: Shae compares noncanonical and canonical displayed graphs of sine and arcsine.

Shae: Looking at this [Fig. 7.6], I would assume they're meaning sine of negative, sorry, one x equals y [*writing* $\sin^{-1}(x) = y$], where x is their vertical distance and y is their angle measure. So the student, they're both [*pointing at both* Fig. 7.3 *and* Fig. 7.6] representing the same thing just considering their outputs and inputs differently.

Int.: So could you say a little bit more about

Shae: Yeah. So they both kept x the horizontal and y the vertical. But, so here [*referring to* Fig. 7.6] their y's show the angle measure and the x's show the vertical distance. So for the inverse sine their input is vertical distance, output is angle measure. And they're showing the same thing here [*referring to* Fig. 7.3], where their input is the vertical distance, which is their y, and their output is the angle measure, which is their x.

[*Shae uses an input value of 1 to argue that both displayed graphs have the same input and output values relative to her respectively defined input and output axes. The interviewer then asks how she would convince a skeptical student who claims that the graphs look different.*]

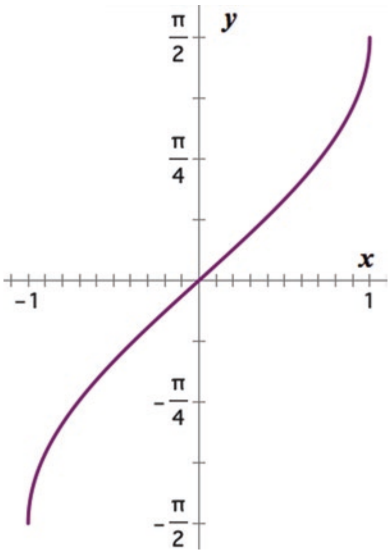


Fig. 7.6 Canonical displayed graph of the arcsine function

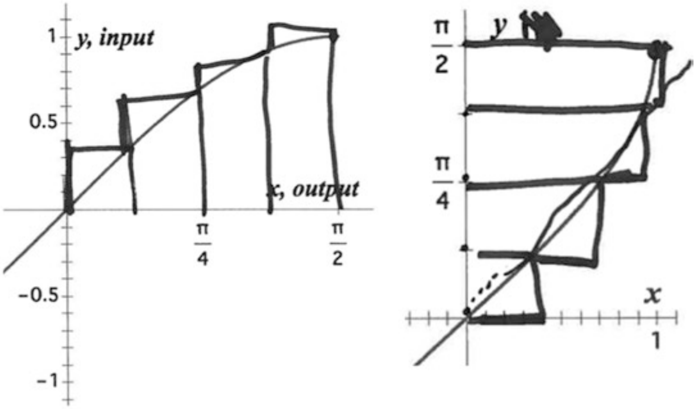


Fig. 7.7 Equivalent conceptions of two displayed graphs

Shae: Oh, you could show the increasing, right. So I mean you could just like disregard the *y* and *x* for a minute, and just look at, like, angle measures. So it's like here [*referring to* Fig. 7.6], with equal changes of angle measures [*denoting equal changes along the vertical axis*] my vertical distance is increasing at a decreasing rate [*tracing curve*]. And then show them here [*referring to* Fig. 7.3] it's doing the exact same thing. With equal changes of angle measures [*denoting equal changes along the horizontal axis*] my vertical distance is increasing at a decreasing rate [*tracing curve*].

Int.: OK.

Shae: So even though the curves, like, this one looks like it's concave up [*referring to* Fig. 7.6 *from* $0 < x < 1$] and this one concave down [*referring to* Fig. 7.3 *from* $0 < x < \pi/2$], it's still showing the same thing. [*Shae denotes equivalent changes on* Fig. 7.3 *and* Fig. 7.6 *as shown in* Fig. 7.7]

Shae's actions indicate her previous learning experiences having resulted in associating a function name with a particular covariational relationship. Furthermore, such a covariational relationship as not constrained to a unique graph or "look," nor was it constrained to a unique function name. Thus, by envisioning each graph to entail some quantity increasing by decreasing amounts as another quantity increases in successive equal amounts, she was able to perceive each graph as mathematically equivalent despite their perceptual differences (e.g., "concave up" versus "concave down"; Fig. 7.7). Mathematical attributes were both properties of Shae's graphs' emergence and the learned function names [$(u, v) \mid -\pi/2 \leq u \leq \pi/2$, $v = \sin(u)$, $u = \arcsin(v)$], and these learned and reconstructed properties formed the basis for her relating the present task to her previous experiences.

7.4 But What of Development?

Recall that two questions generated by the opening vignettes were:

1. In what ways do students' graphical shape thinking influence their construction of relations of similarity between previous and current learning experiences (i.e., transfer)?
2. Relatedly, in what ways do students' attempted construction of relations of similarity between previous and current learning experiences (i.e., transfer) influence their development of graphical shape thinking?

Regarding the first question, in the case of static graphical shape thinking, indexical associations based on perceptual features form the basis for constructing relations of similarity, supporting students in assimilating those contexts in which figurative aspects of shape prove viable. For instance, when experiencing a novel graph in some coordinate system, students recently completing an instructional sequence emphasizing static graphical shape thinking might anticipate and impose perceptual and figurative features of shape on that novel graph (see Lobato et al., 2012). In the case of emergent graphical shape thinking, the logico-mathematical operations of quantitative and covariational reasoning form the basis for constructing relations of similarity, supporting students in assimilating those contexts in which those covariational and quantitative schemes prove viable. Students recently completing an instructional sequence emphasizing emergent graphical shape thinking might anticipate and impose covariational properties on some novel graph (see Moore et al., 2013).

Whereas the first question is focused on how students' prior learning experiences influence their present experience, the second question opens a focus on explaining the ways in which students' transfer actions can, in turn, result in modifications to those meanings constructed during previous learning experiences. With respect to graphical shape thinking, I contend that sequential processes of forward and backward transfer occasion reciprocal acts of assimilation and accommodation that

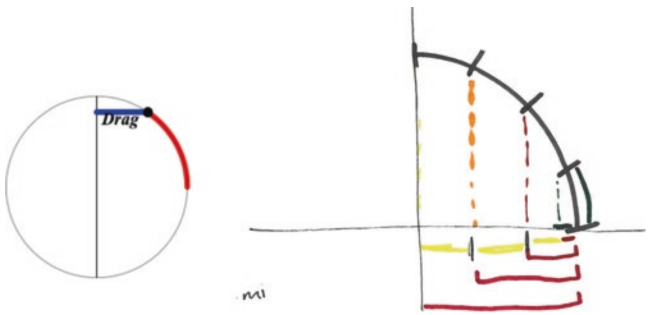


Fig. 7.8 An example of partitioning activity to show horizontal segments decreasing by increasing amounts for successive equal variations in arc (Stevens & Moore, 2017, p. 712)

provide the basis for constructing abstracted meanings rooted in emergent shape thinking.

To illustrate, I draw on a synthesis of a student's actions when prompted to construct a graph representing how two quantities vary together in the context of circular motion (see Stevens & Moore, 2017, for a more detailed account of the student's actions). What follows occurred after a group session in which the student, Lydia, and two other students engaged in partitioning activities (e.g., Fig. 7.8) with a diagram of a circle to identify and reason about variations in horizontal or vertical distance from the vertical or horizontal diameter, respectively, for equal variations in arc length (i.e. the sine and cosine relationships).

After the group session, we engaged Lydia in an individual session. We asked her to return to the circular motion context to gain insights into how her experiences during the group session might have influenced her reasoning. She first constructed variations in horizontal distance for equal changes in arc length. She appropriately concluded that the horizontal distance decreased by an increasing magnitude for an equal change in arc length as the point rotated from the start to the 12 o'clock position (consistent with Fig. 7.8). Her actions and claims were consistent with the group conclusions from the previous session.

We then asked Lydia to create a graph representing this relationship (i.e., the normative Cartesian graph for the cosine relationship), again attempting to gain insights into how the group sessions influenced her thinking as well as how she drew relations of similarity between circle and graphical contexts. Lydia immediately drew a curve that perpetually resembled the normative Cartesian graph for the sine relationship (Fig. 7.9, bottom, with only the axes and curve).

What occurred next was an interaction in which Lydia attempted to engage in compatible physical actions with the circle context and her drawn curve while

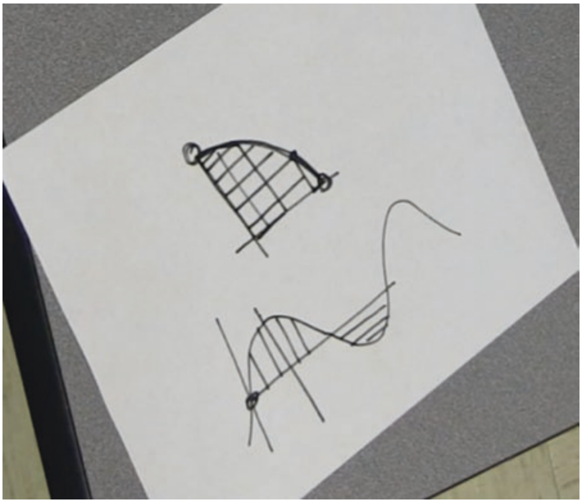


Fig. 7.9 Lydia's drawn graph and circle context (Stevens & Moore, 2017, p. 713)

simultaneously constructing the same covariational properties. Namely, she attempted to:

- Partition along an arc (the circle in the circle context and the curve in the Cartesian context)
- Draw horizontal segments
- Draw vertical segments
- Identify segments that were increasing by decreasing amounts for equal, successive variations in arc length, a property consistent with the sine relationship
- Identify segments that were decreasing by increasing amounts for equal, successive variations in arc length, a property consistent with the cosine relationship
- Draw and identify equal changes along the horizontal Cartesian axis, which was an action done repeatedly in the group session and class in which she was enrolled

Attempting to construct and identify all of these in both the graphical and circle contexts perturbed Lydia. After several different attempts and a sustained period of time, she explained, "I like see the relationship, and I can explain it to a point, and then I get like—I confuse myself with the amount of information I know about a circle that I was just given to me by a teacher, and then what I've like discovered here [*referring to the teaching sessions*]."

Recall that the group session included a focus on both the sine and cosine relationships. An explanation for Lydia's perturbation is that she expected all of the actions and properties of both to be relevant to both the circle and her graph. Thus, in her attempt to relate her current activity to that in the group sessions, she conflated particular figurative and perceptual features of her and her classmates' actions (i.e., static graphical shape thinking) and those quantitative and covariational conclusions those actions and their results indicated (i.e., emergent graphical shape thinking). This left her unable to relate the present experience, the group session outcomes, and her previous instructional experiences to her satisfaction.[6]

I interpreted Lydia's actions to indicate both elements of static graphical shape thinking and emergent shape thinking, and her conflating these elements constrained her ability to relate the group session to her present experience. I thus decided to engage Lydia in another sustained round of interactions so that she could further reflect on her activity and those actions she attempted to transfer from the group session. I also drew her attention to identifying the quantities of the circle context, illustrating several particular values of those quantities in the circle context, identifying how those values related to her graph, and repeating this process (see Fig. 7.10

[6] It is important to note that a traditional transfer perspective would frame Lydia as not transferring her knowledge from the group sessions because of her not successfully completing the problem in ways aligning with researcher intentions. The AOT perspective, however, allows for a much more nuanced and productive account of Lydia's transfer actions because of its sensitivity to transfer from her viewpoint. Lydia was transferring actions from the group session, and far too many to establish a personal state of understanding.

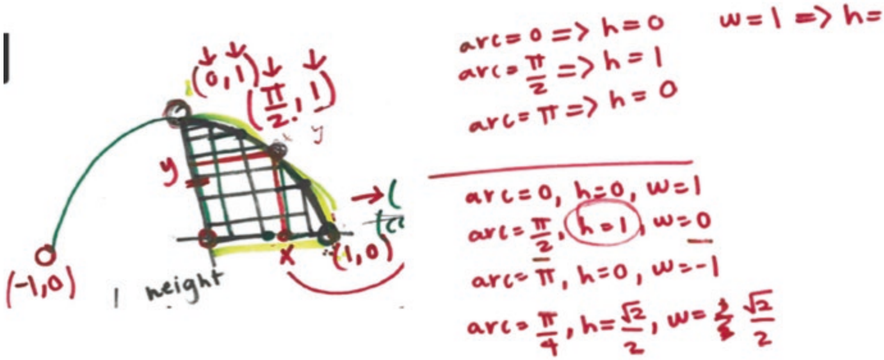


Fig. 7.10 Lydia's annotated diagram of identifying quantities and values (Stevens & Moore, 2017, p. 714)

for her work with the diagram). It was during this process that Lydia had a realization (Excerpt 4).

Excerpt 4: Lydia has a realization (Stevens & Moore, 2017, p. 714, with "{ }" denoting modifications added for clarifying purposes).

Lydia: Because this is my – This is *x* – um, *x-y* plane, then here I'm saying at this point [*the origin*], my width is 0, my arc length is 0, and my height is 0.
Int.: Width is 0, my arc length is 0 and my height is 0.
Lydia: Wait, but then I said {*referring to the situation*} at arc length 0, and [*laughs*] height is 0, then my width should be 1.
Int.: And your width should be 1, right? What about at pi-halves? What should we have?
Lydia: Then I should have a height of 1 [*pointing to curve for an abscissa value of pi/2*].
Int.: Okay.
Lydia: And then my width should be 0 {*focus remaining on her graph*}. So this graph does not do anything with the *x-y* plane.

[*Lydia summarizes this claim and then the researcher asks Lydia to consider an arc length of pi radians.*]

Lydia: Then my arc length on the *x*-axis [*motions across horizontal axis*] should be pi. My height should be 1 – or 0, and then my *x*-value should be negative 1. So this [*referring to her drawn graph*] just doesn't have – then this doesn't relate to the *x*, the width [*referring to width from the situation*], just this graph. So my whole circle talks about width and height and arc, but then this graph itself only talks about arc and height. [*speaking emphatically*] Done it. [*laughs*]

{Lydia then reasons emergently about her graph.}

The beginning of this interaction continues to illustrate the influence of the group session on Lydia's activity. Namely, Lydia continued her attempt to incorporate each of the three relevant quantities and their corresponding contextual segment orientations into her drawn graph. In this case, however, she exhibits a more explicit focus on quantities' values so that figurative actions were subordinate to quantitative operations. In doing so, Lydia had a realization about the outcome from the group session; she came to conceive her drawn graph as an emergent trace of two particular quantities—arc length and height—in a way compatible with the circle context. Notably, Lydia indicated that this was a pivotal moment for her (e.g., "Done it"), and her developing emergent graphical shape thinking as a way to relate a context and graph became a meaning she transferred forward for the remainder of the study (Stevens & Moore, 2017).

Reflecting on Lydia's progression, I underscore that her initial actions in the circle context were stable and such that we interpreted her to have reasoned quantitatively and covariationally. It was in the act of transferring those actions to her recollection of the drawn graph from the group session (i.e., forward transfer) that she was perturbed. It was then through several processes of reconstructing and relating her actions in the present contexts and from the group sessions that she was able to identify and isolate those actions critical to her (and her group's) activity and those that were merely a product of the representational system. More broadly, Lydia's actions highlight the potential affordances of sequential processes of forward and backward transfer in the context of representational activity and graphical shape thinking. Namely, when a student experiences the opportunity to construct and represent a particular relationship in multiple and varied ways across multiple learning experiences, they are afforded the opportunity to identify and differentiate between those (physical and mental) actions associated with emergent graphical shape thinking so that only vestiges of figurative activity remain. Both Thompson (1994b) and Lobato and Bowers (2000) identified that such an opportunity is the underlying foundation to a productive view of multiple representations.

Before closing, I note that when speaking of constructing and representing a relationship in multiple and varied ways, I am referring to a multitude of contexts that permit anticipating and enacting quantitative operations on available figurative material. For instance, event phenomenon and coordinate systems (e.g., a Ferris wheel ride, a bottle filling with water, and the polar coordinate system) permit quantitative operations on figurative material associated with quantities (e.g., a traversed arc length, a segment representing the height of water, and a directed angle measure and radial distance). If event phenomenon, multiple coordinate-system orientations, and multiple coordinate systems are used in tandem, it provides students a plethora of opportunities to differentiate quantitative operations from figurative forms of action (Moore, Stevens, Paoletti, Hobson, & Liang, 2019). In contrast, tables, formulas and written phrases—each a representation—do not entail figurative material that permit quantitative operations. I do not downplay the important use of tables, formulas, and phrases, but rather highlight the difference in their use as compared to that of event phenomenon and graphs as it relates to affording students the opportunity to simultaneously engage in and differentiate between quantitative operations and figurative forms of action.

7.5 Moving Forward

Von Glasersfeld (1982) defined a *concept* as "any structure that has been abstracted from the process of experiential construction as recurrently usable" (p. 194). The term *abstraction* has a long history in mathematics education, and the term accordingly is met with far too many different interpretations and perspectives to describe and synthesize here (e.g., Dubinsky, 1991; Piaget, 2001; Sfard, 1992; Simon et al., 2010; von Glasersfeld, 1991; Wagner, 2010). As a concise and simplistic definition for operational purposes, abstraction is the process of becoming consciously aware of and differentiating between one's actions (physical and mental) that are critical to some conceived concept and those that are not (Moore, Stevens, et al., 2019; Piaget, 2001). As Wagner (2010) explained, abstraction is not a decontextualizing process that results in constructing something devoid of context, but rather, an abstracted concept becomes more sensitive to both the similarities and differences among perceived contextual instantiations of the concept.

Lydia's actions illustrate such a process of abstraction in her differentiating between those actions and operations that are quantitative and covariational in nature and those that are a product of representational conventions and figurative aspects of a context perceived as entailing that relationship (Moore, Stevens, et al., 2019). In doing so, Lydia eventually constructed a meaning for graphing—emergent graphical shape thinking—that consisted of a covariational structure she could describe as if it is independent of the specific figurative material associated with a context. She could also transfer this way of thinking to assimilate novel contexts or situations permitting the operations constituting her way of thinking. It is in this way that her thinking became abstract, that she constructed a concept; she constructed a structure so that its mathematical properties and actions were anticipated independent of any particular instantiation of them, thus not being tied to any particular two quantities and associated context.

Lydia represents only one case, and it remains to be seen how students' learning can be supported through sequential processes of forward and backward transfer in the context of repeated and varied opportunities to construct and represent covariational relationships. Much is left to understand about the initial and ongoing development of graphical shape thinking, especially in the context of students who are experiencing graphing for their first time. The forms of graphical shape thinking do not currently represent developmental stages, nor are the graphical shape thinking constructs as predictive and explanatory as those in areas like units coordination (Steffe & Olive, 2010). To make a claim of developmental stages requires research focused on students' persistence in using them as ways of thinking and evidence that their current schemes impede their thinking at a higher level, and thus research along those lines is a necessary and important next step of research. Furthermore, a current limitation of shape thinking and its forms is that they stem from working primarily with secondary students, undergraduate students, and postgraduate students (i.e., teachers). Detailed insights regarding the initial development of students' meanings for graphs as related to graphical shape thinking are thus needed, especially during students' formative years of constructing displayed graphs. Importantly, there is evidence suggesting that emergent graphical shape thinking is a readily accessible meaning for middle-grades and secondary students (Ellis, 2011; Ellis et al., 2015; Johnson, 2012, 2015).

One productive future line of inquiry will be investigating students' meanings for graphs in the context of some topic, such as students' derivative or rate of change meanings. Researchers taking a topical focus will contribute nuanced descriptions of the schemes and operations that comprise the forms of shape thinking and are specific to those topics (e.g., conceiving a displayed graph as relating multiplicative and additive structures, Ellis et al., 2015). Additionally, researchers that take a topical approach can gain insights into the extent that the forms of shape thinking enable productive transfer as it relates to learning those topics. A complementary line of inquiry to a topical focus will be investigating students' meanings across multiple contexts and topics. Researchers who consider shape thinking and its forms across multiple contexts and topics will have opportunities to make generalizations with respect to students' meanings and transfer.

Another productive future line of inquiry will be characterizing relationships between students' graphical shape thinking, backward transfer, and their learning. At the prospective and practicing teacher level, there is evidence suggesting their meanings not only foreground static graphical shape thinking (Thompson et al., 2017), but that their meanings can conflict with reasoning emergently (Moore, Stevens, et al., 2019). Moore, Stevens, et al. (2019) specifically illustrated that prospective teachers can produce graphs emergently that differ from those produced statically, especially under noncanonical coordinate orientations. In such cases, the prospective teachers experienced a perturbation. Although not the focus of the authors' study, their findings suggest the potential for backward transfer. When perturbed as a consequence of reasoning emergently, the prospective teachers showed evidence of reflecting on and beginning to analyze their previously constructed meanings, which had been consistent with reasoning statically. These initial acts of perturbation and reflection can be the genesis of backward transfer (Hohensee, 2014; Lobato & Siebert, 2002), and future researchers should explore the affordances of these situations in promoting productive backward transfer.

In closing, I make an instructional and curricular comment for both educators and researchers. Lobato et al. (2012) convincingly illustrated how numerous classroom factors can influence students' propensity to construct meanings consistent with emergent or static graphical shape thinking. Complicating the matter, researchers have provided results working with teachers and students that suggest emergent graphical shape thinking is not currently a widely held learning goal in classrooms (Carlson et al., 2002; Thompson, 2013b; Thompson et al., 2017). Thus, it will take concerted and intentional efforts, both inside and outside the classroom, if emergent graphical shape thinking is to become a targeted learning goal of mathematics educators. Specific to curricular materials, I view typical K-16 textbooks and curricula to be nearly devoid of intentional or sustained efforts to engender and support emergent graphical shape thinking. At best, textbooks and curricular narratives sustain a focus on displayed graphs as consisting of coordinate pairs and states of values, which is not equivalent to a focus on covariation, magnitudes, or a displayed graph's emergence and is especially problematic when combined with examples like those above that treat displayed graphs statically (Carlson et al., 2002; Frank & Thompson, 2019; Thompson & Carlson, 2017; Thompson et al., 2017). Based on this observation, I find an important area of work to be the design of curriculum and instructional experiences that target students' emergent shape thinking. More specifically, I perceive a need for instructional activities and interactions in which it is productive for students to differentiate between mathematical properties necessary to all graphs of a relationship and those properties that are a consequence of the conventions of a coordinate system.

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