Breaking Conventions to Support Quantitative Reasoning

Kevin C. Moore
University of Georgia

Jason Silverman
Drexel University

Teo Paoletti & Kevin R. LaForest
University of Georgia

Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. R. (2014). Breaking conventions to support quantitative reasoning. *Mathematics Teacher Educator, 2*(2), 141-157.

Available at: http://www.nctm.org/publications/article.aspx?id=40724

Breaking Conventions to Support Quantitative Reasoning

Kevin C. Moore
University of Georgia

Jason Silverman
Drexel University

Teo Paoletti and Kevin LaForest
University of Georgia

Quantitative reasoning is critical to developing understandings of function that are important for sustained success in mathematics. Unfortunately, preservice teachers often do not receive sufficient quantitative reasoning experiences during their schooling. In this paper, we illustrate consequences of underdeveloped quantitative reasoning abilities against the backdrop of central function concepts. We also illustrate tasks that can perturb preservice teachers' thinking in ways that produce opportunities for quantitative reasoning. By implementing strategically designed tasks, teacher educators can support preservice teachers—and students in general—in advancing their quantitative reasoning abilities and their understanding of secondary mathematics content.

Key words: Function; Prospective secondary teachers; Quantitative reasoning; Rate of change

The authors of education policy documents and panel reports have highlighted the central role that function can play in school mathematics, specifically calling for teachers to take a function-based approach to the teaching and learning of middle school and secondary mathematics (Common Core State Standards Initiative, 2010; National Mathematics Advisory Panel, 2008; RAND Mathematics Study Panel, 2003). In examining the Common Core State Standards for Mathematics (CCSSM), the traditional definition of function (a rule that assigns exactly one output to each input) first appears as a content focus for Grade 8. Throughout the remainder of the standards, the CCSSM focuses on function more broadly: interpreting and building functions, modeling with functions, and engaging in mathematical practices associated with function. In the CCSSM, the study of function as *relationships between covarying quantities* is envisioned as a unifying theme for middle and secondary mathematics.

Incorporating the CCSSM Mathematical Practice of *reasoning quantitatively* (or equivalently *quantitative reasoning*) into the teaching of function is fitting, given that researchers (e.g., Leinhardt, Zaslavsky, & Stein, 1990; Oehrtman, Carlson, & Thompson, 2008) have argued that students' difficulties with the function concept often stem from limited opportunities for quantitative reasoning. A vision for a quantitative reasoning approach to function also raises a significant question for mathematics educators and mathematics teacher educators. If neither students nor teachers are receiving sufficient opportunities to develop their ability to reason about relationships between quantities (Moore & Carlson, 2012; Oehrtman et al., 2008; Smith III & Thompson, 2008), should we expect teachers to teach for these same understandings and reasoning abilities?

Obviously, the answer to this question is "No, not without help," and interventions are needed to support connections between function and quantitative reasoning. We work toward addressing this need by illustrating quantitative reasoning and its connection to various aspects of function. Namely, we show how certain mathematical conventions can confound preservice teachers' (PSTs') function understandings and quantitative reasoning. In presenting the data, we illustrate several tasks that can be used to support PSTs' development of quantitative reasoning in ways that are foundational to the function concept. We conclude with suggestions for teacher educators looking to incorporate quantitative reasoning into their work with PSTs.[1]

Function, the Vertical Line Test, and Quantitative Reasoning

Although there is widespread agreement that function forms a core content area for college and career readiness, students often view function as little more than "plugging in" numbers, executing calculations, and applying the vertical line test. Because such a focus is

1 Although the current article reports on work with PSTs, we have also implemented the ideas and tasks here with in-service teachers and students with equal success. Thus, when referring to work with PSTs, we could just as easily be referring to work with in-service teachers or students.

conceptually divorced from core ideas of function and the CCSSM vision, students face persistent and widespread difficulties with the function concept that inhibit their success in secondary and undergraduate mathematics. One explanation for these persistent difficulties is that the teaching of function often lacks a focus on reasoning about quantities that vary in tandem (Lobato & Bowers, 2000; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994). Without a focus on quantities and their relationships, students do not relate functions to the situations that they model, instead relying on procedural understandings.

To illustrate, consider a procedure that students often rely on: the *vertical line test*. Although this procedure enables students to provide a correct response in particular situations, difficulties arise when students' use of the vertical line test does not include an awareness of the conventions that underlie the procedure. For example, only 19% of the students in a study by Breidenbach, Dubinsky, Hawks, and Nichols (1992) were able to describe the graphed relationship in Figure 1 as a function; the students' reliance on the vertical line test inhibited their ability to consider x as a function of y. Likewise, Montiel, Vidakovic, and Kabael (2008) noted that students relied on the vertical line test when considering relationships in the polar coordinate system. Despite the change of coordinate system, students applied the vertical line test to conclude that the graph of $r = 2$ does not represent a function (Figure 2).

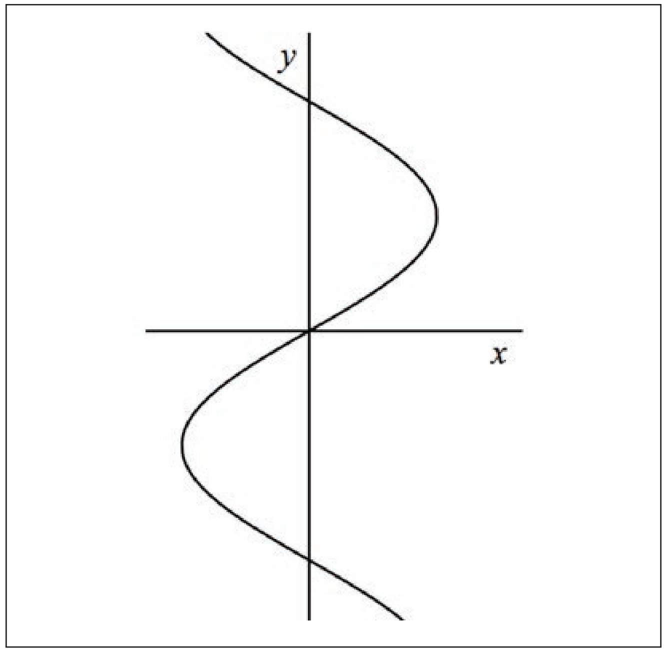

Figure 1. **Reproduced graph from Breidenbach et al. (1992, p. 281).**

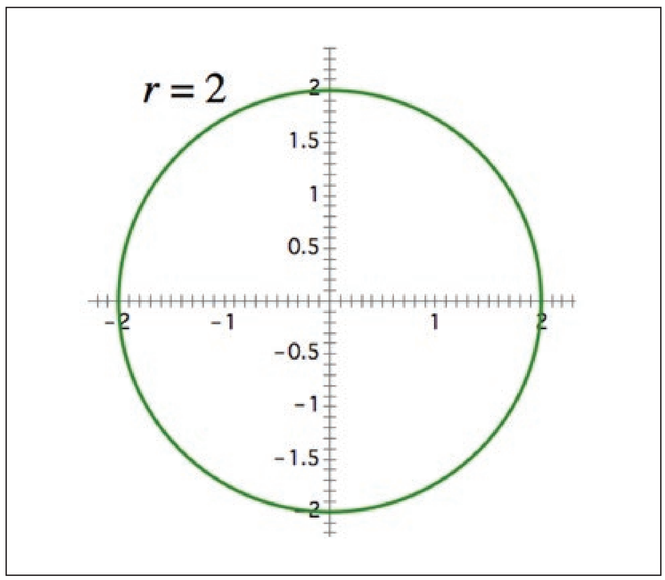

Figure 2. **Reproduced graph from Montiel et al. (2008, p. 70).**

We argue that cases such as these, where an individual's understanding of function was limited by a reliance on the vertical line test, exhibit the consequences of function understandings that do not entail quantitative reasoning. Quantitative reasoning—conceiving of a situation, constructing quantities (attributes of objects that can be measured using a formal or informal measurement process), and reasoning about relationships between the quantities—has received increased emphasis in mathematics education research and policy. Included as part of the Common Core State Standards for Mathematical Practice, "quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects" (Common Core State Standards Initiative, 2010, p. 6). One primary distinction between quantitative reasoning and typical "numerical" reasoning is that quantitative reasoning does not necessarily involve numerical values. For example, one can reason about the relationship between the possible length and width of the sides of a rectangle with a fixed perimeter without knowing the perimeter or particular values of length and width.

Returning to Figure 1, and in contrast to typical descriptions that focus first on determining whether a particular relationship property holds for that relationship (e.g., each value of the input quantity corresponds to a unique value of the output quantity),[2] a quantitative reasoning-based description involves *first* understanding the graph as representing corresponding quantities' values that covary simultaneously (i.e., "creating a coherent representation of the problem at hand"). Next, determining whether the relationship between these paired values is a function involves (a) choosing independent and dependent variables with the understanding that either quantity can be denoted as the independent variable and (b) determining if a particular relationship property holds in the chosen dependency. With respect to Figure 1, we conclude that x is a function of y and y is not a function of x; the graph is and is not a representation of a function all at once.[3]

Our intention is not to denounce the vertical line test. Instead, we use the vertical line test to draw attention to the different understandings that individuals might hold when using such a procedure. In one case, an individual might view the vertical line and graph solely in terms of counting how many times two "wires" in the plane meet. On the other hand, an individual might apply the vertical line test while holding in mind an underlying process that includes: (a) an awareness of how two quantities are represented in the given coordinate system; (b) defining an axis (i.e., the horizontal axis) as the input quantity; and (c) determining how many output values correspond to each input value. The former understanding does not entail quantitative reasoning, instead focusing on perceptual features of the graph, which can lead to students misapplying the procedure. The latter understanding has a basis in quantitative reasoning and thus has a greater potential to be generalizable to other situations (Ellis, 2007).

Although we believe that understandings entailing quantitative reasoning are enduring and increase the likelihood that students will analyze relationships and determine function*ness* correctly in a variety of contexts, developing such understandings is not trivial. Individuals need repeated opportunities to reason quantitatively in the context of major ideas like function so that they can construct knowledge transferrable to novel situations (Lobato, Rhodehamel, & Hohensee, 2012). In what follows, we illustrate tasks that give PSTs repeated opportunities to engage in quantitative reasoning in ways that can perturb and advance their understandings. We also focus on tasks that provide educators an opportunity to gain insights into PSTs' quantitative reasoning.

The Study

The data used in this article is drawn from clinical interviews (Clement, 2000; Goldin, 2000) and a classroom teaching experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Steffe & Thompson, 2000). The participants were 17 PSTs enrolled in a content-focused course as part of a secondary mathematics education program. All PSTs participated in the classroom teaching experiment, and 9 of the PSTs volunteered for the clinical interviews. The goal of the study was to characterize PSTs' quantitative reasoning upon entering the course and determine ways that their quantitative reasoning was supported during the course.

Participants and Course

The participants were predominantly 3rd-year undergraduate students (in credits taken) at a large public university who were taking the first pair of courses in the secondary program. The 4-semester program entails 3 semesters with a content-focused course, a pedagogy-focused course, and a field experience; the 4th semester is student teaching. Admission to the program requires that the PSTs have successfully completed an undergraduate calculus sequence and at least one mathematics course past this sequence (e.g., linear algebra or differential equations).

The lead author, who served as the course instructor, designed the first content-focused course to engage the PSTs in quantitative reasoning (Smith III & Thompson, 2008) and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) in the context of secondary mathematics topics. These topics include angle measure, trigonometric functions, polar coordinates, exponential functions, linear functions, and rate of change. The lead author drew on his research experiences in quantitative reasoning and trigonometry (Moore, 2012, 2013; Moore & Carlson, 2012; Moore & LaForest, in press) when designing the course.

The instructor conducted class sessions in a way that engaged the PSTs in problem solving (Carlson & Bloom, 2005) and productive mathematical discourse (Clark, Moore, & Carlson, 2008). During class sessions, the PSTs predominantly worked in groups of three to four on open-ended tasks. The instructor and two teaching assis-

2 For the purpose of this paper, we consider single-valued functions, as opposed to multi-valued functions. Single-valued functions are such that each element of the domain is associated with a unique element of the range. Multi-valued functions are such that each element of the domain is associated with one or more elements of the range.

3 We note that this speaks to a loaded question often asked to students: Is the graph a function? Without presuming particular conventions, this question is ambiguous. What is really being asked of students is typically "Is y a function of x?"

tants observed the PSTs and pushed them to give mathematical explanations that referenced quantities and their relationships as opposed to describing calculations and answers. Group members shared a 2′ x 3′ whiteboard to record their work. When groups reported their work, the lead instructor typically chose groups that obtained different solutions in order to draw attention to and discuss these differences.

Clinical Interviews

In order to understand PSTs' quantitative reasoning upon entering the course, we conducted clinical interviews (Clement, 2000; Goldin, 2000) with 9 PSTs at the beginning of the semester. Each interview lasted 90–120 minutes. All interviews were videotaped and transcribed. The data was then analyzed using conceptual analysis techniques (von Glasersfeld, 1995) in an attempt to build viable models of the PSTs' thinking by collectively developing, testing, and refining conjectures about their thinking. The research group, which consisted of the author team and those contributing members named in the acknowledgements, met throughout the interview and analysis process in order to discuss observations and data analysis.

Teaching Experiment

Drawing on the clinical interview outcomes, we conducted a teaching experiment that consisted of fifteen 75-minute teaching sessions. Teaching experiments differ from clinical interviews primarily in their purpose. Whereas clinical interviews intend to explore an individual's thinking without promoting shifts, teaching experiments intend to engender and characterize fundamental shifts in an individual's thinking (Steffe & Thompson, 2000). Reflecting this purpose, we used a teaching experiment to identify boundaries in the PSTs' thinking and determine how to support the PSTs in engaging in quantitative reasoning in order to overcome these boundaries.

Data collection involved videotaping and digitizing the PSTs' activity during both group work and whole-class discussions. Also, the research team observed each teaching session, taking notes of the interactions between the researchers and PSTs. We debriefed immediately after each session in order to discuss the PSTs' thinking, document all instructional decisions, and plan for the subsequent sessions. Like the clinical interviews, our retrospective analysis of the data involved performing a conceptual analysis of the data. Previous works (Moore, 2013; Moore, Paoletti, & Musgrave, 2013) provide a more detailed description of the use of this methodology and analysis technique.

Gaining Insights Into PSTs' Quantitative Reasoning

PSTs have constructed stable understandings over years of coursework that enable them to solve a wide variety of problems. However, their activity is often procedural in nature, which does not necessarily give insights into their quantitative reasoning; a PST providing a procedural solution *does not* imply that she cannot reason quantitatively. In order to draw PSTs' quantitative reasoning (or lack thereof) to the surface, we designed tasks that did not follow common school conventions (e.g., graphing in a situation with x on the vertical axis and y on the horizontal axis). We conjectured that PSTs with underdeveloped quantitative reasoning abilities would experience difficulties handling these situations, as the departure from conventions inhibits the use of procedures based on these conventions. In contrast, we believed that PSTs who had more developed quantitative reasoning abilities would provide coherent explanations by focusing on relationships between quantities. Either way, their responses provided insight into what extent they relied on the analysis of quantities and relationships.

Interview Task 1

Figure 3 presents one clinical interview task, which was posed in two parts. First, we presented the graph with unlabeled axes (Figure 3a) and a prompt explaining that this was the completed work of a middle school student asked to graph $y = 3x$. After the PST provided conjectures about how the student might have been thinking when creating the graph, the PST was then presented with labeled axes (Figure 3b) and a prompt explaining that the student, when questioned about her solution, added the labels. The PST was again asked to conjecture how the student was thinking.

In response to the first prompt, each of the 9 PSTs provided viable explanations for the student's solution. Five of the PSTs conjectured that the student either misunderstood the slope (e.g., going "over three" and "up one" rather than "over one" and "up three") or graphed the relationship $x = 3y$. In addition to providing explanations regarding a misunderstanding of slope or graphing $x = 3y$, 3 of the remaining PSTs also conjectured that the student may have graphed y on the horizontal axis. The 9th PST only conjectured that the student had graphed y on the horizontal axis.

Whereas the PSTs did not find the slope and $x = 3y$ explanations to be too troublesome, the possible solution of the student graphing y on the horizontal axis did cause

Figure 3. **The two posed graphs for Interview Task 1.**

some perturbation. As an example, consider Bonnie's explanation when presented with Figure 3b (Excerpt 1).

As the interaction continued after Excerpt 1, Bonnie maintained that she was unable to consider the posed graph as representing $y = 3x$. An explanation for her inability to conceive the graph as $y = 3x$ is that her interpretation of the student's solution did not entail quantitative reasoning. Rather, Bonnie relied on recalling slope-graph pairs and how a graph should visually appear (e.g., a left-to-right upward sloping line means a positive slope). Because the second graph was not in a conventional form, her memorized slope-graph pairs were not applicable without rotating the graph, and even then she judged her rotated graph as "still wrong" because she interpreted the rotated graph to have a negative slope.

Collectively, 7 of the interviewed PSTs remained unable to evaluate the student's solution, claimed that the solution was wrong, or stated that the solution gave them a sense of discomfort. Even those PSTs who considered specified x and y values to conclude Figure 3b is a graph of $y = 3x$ expressed uneasiness with the validity of the solution because the line did not appear to have the cor-

Excerpt 1

Bonnie discussing student's solution to $y = 3x$ task

Int: How might the student be thinking about this?

Bonnie: Umm . . . like this [*spinning paper 90 degrees counterclockwise*]. [*Laughs*]. Like, because if you turn it this way then this [*traces left to right along the x-axis, which is now in the horizontal position*] and this [*traces top to bottom along the y-axis*] and it would be still not right though [*spinning paper back to original orientation*].

Int.: And how would you respond to this student if they said, "Well, here's [*pointing to the graph in original orientation*] how I'm thinking?"

Bonnie: . . . I mean, the only way I can think of it is like this [*spinning paper 90 degrees counterclockwise*], and it's still wrong because this [*laying the marker on the line, which is now sloping downward left to right*] is negative slope. So I would just, I would just explain to them, like the difference between the *x*- and *y*-axes and umm . . . show them like the difference between positive and negative slopes also. Because that's something that, like, when I was in middle school we, like, learned kind of like a trick to remember positive, negative, no slope, and zero [*making hand motions to indicate each*]. Like where the slopes were. And it's stuck with me until now, so it's important to know which direction they're going . . .

rect slope.[4] Such difficulties and uneasiness on the part of the PSTs illustrate the consequences of slope understandings that do not entail or foreground quantitative reasoning. From a quantitative reasoning perspective, *slope* means the rate of change between two covarying quantities. Hence, rotating a graph or changing the axis orientation does not change the slope or relationship; the rate of change of y with respect to x is 3 regardless of the orientation (Figure 4). But for PSTs like Bonnie who hold slope understandings restricted to perceptual cues, rotating a presented graph *does* change the relationship, as the rotation yields a different visual picture. For instance, because Figure 4b is a line sloping downward left to right, Bonnie considered the linear function to have a negative slope, thus making it impossible for it to be a graph of $y = 3x$.

Interview Task 2

As an example of another clinical interview task, we showed the PSTs two graphs that presented the same distance and time relationship but with different axis orientations (Figure 5). We gave the PSTs the following prompts: (a) Are these graphs the same or different? and (b) Determine whether each graph represents a function.

We designed the task to be problematic to a PST relying on the vertical line test: both graphs represent the same distance–time relationship, yet based on the vertical line test only one graph represents a function. As Excerpt 2 illustrates, the PSTs were not satisfied with this conclusion.

Monica's confusion illustrates the conflict that can result from a reliance on the vertical line test. Monica recognized that both graphs convey the same distance–time relationship and also found it problematic to claim that only one graph represents a function. However, because only one graph passed the vertical line test, she was unsure about whether Figure 5a represents a function, and the interaction ended with her unable to reconcile this perturbation.

Of the 9 PSTs who discussed whether Figure 5a (or a graph like it) represents a function, 7 arrived at a conclusion similar to that reached by Monica. Six of these 7 PSTs maintained that a graph like Figure 5a does not represent a function because of the vertical line test or because inputs of a function *must* be represented on the

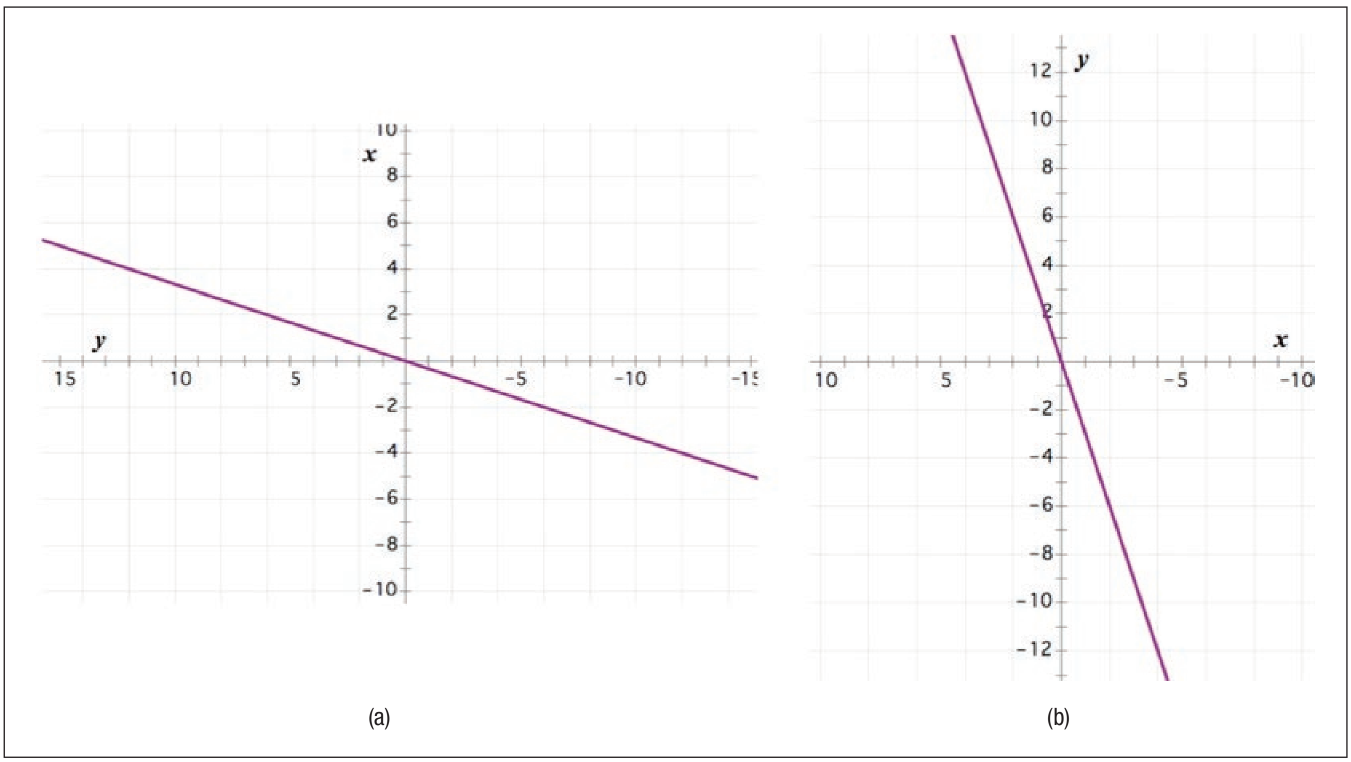

Figure 4. **Two graphs of $y = 3x$, each with the same rate of change.**

4 It is interesting to note that of the PSTs who deemed Figures 3a-b to be correct if the y axis was denoted horizontally, a majority preferred a student who incorrectly went "over three" and "up one" to a student who used unconventional axes but graphed a correct relationship. In a way, these PSTs viewed using an unconventional orientation as wrong—to them, the convention was *not* a convention, it was a rule that the student needed to follow in absolute.

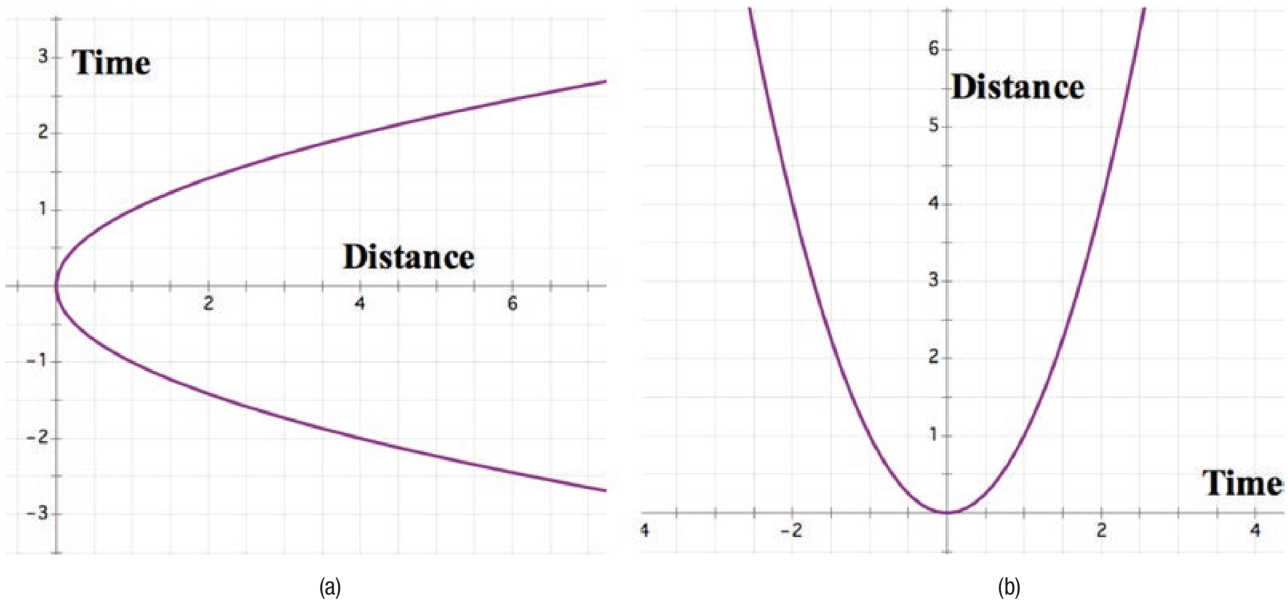

Figure 5. **Two graphs, one relationship.**

Excerpt 2

Monica unable to resolve her conflict

Prior to the following exchange, Monica had used the vertical line test to determine that Figure 5b is representative of a function while Figure 5a is not. After Monica identified that for each time there was an associated unique distance in Figure 5a, which she took to contradict her conclusion drawn from the vertical line test, she expressed that she was no longer sure how to answer the question and looked to the interviewer for assistance. The following interaction then occurred.

Int.: So what do you think, you're still saying not a function on this [*Figure 5a*] one? This one [*Figure 5b*] is, that one's [*Figure 5a*] not? Or are you torn? Where you at, which side are you on?

Monica: Man! I feel like I was so. Like, I feel like no no no no it's [*Figure 5a*] not a function. But like, when you ask me these questions it kinda makes it seem like it is. Cause we're really. It's the same graph. We're talking about the same thing, it's just like one's turned and one's not.

Int.: Is the time–distance relationships between the two—are they the same or different?

Monica: I think they're the same.

Int.: They're the same, right? So it's kinda interesting?

Monica: So why is, why do we do that . . .

Int.: In why—so you're saying also why couldn't we call that [*Figure 5a*] a function?

Monica: Right.

Int.: If that's [*Figure 5b*] a function we should be able to call that [*Figure 5a*] a function?

Monica: That's just—

Int.: That's a good question.

Monica: So can we?

Int.: What do you think? [*Monica laughs*] What do you think, can we?

Monica: Man! I don't, I don't know. I wanna say yes, but no.

Int.: But your background tells you no?

Monica: Yeah. But I think I should say yes. I just feel the background [*referring to the vertical line test*], I feel like the background might be right. [*laughs*] I don't know, this is so weird!

horizontal axis. Like their solutions to Interview Task 1, the PSTs' activity highlights the consequences of understandings that do not entail quantitative reasoning, and particularly how understandings tied to conventions can conflict with attempts to reason quantitatively. From a quantitative reasoning perspective, both graphs represent a relationship such that distance is a function of time and time is not a function of distance; each graph is and is not representative of a function, depending on the chosen dependency. However, a function understanding constrained to the vertical line test or input on the horizontal axis does not foreground relationships between two quantities in such a manner, leaving the PSTs to then question whether the two graphs convey the same relationship and, if so, why one graph is deemed to be the graph of a function and the other graph is not.

Classroom Intervention

In the clinical interviews, the interviewer did not help a PST resolve his or her conflict or confusion. However, the insight gained from these interview tasks (and others like them) provided starting points for the instructional activities used during the teaching experiment. Over the course of the implementation of the instructional activities, we noticed an increase in the PSTs' propensity to reason quantitatively, particularly when initially engaging in a task. We provide a summary of course activities and provide additional example outcomes in the **Appendix**.

In this section we detail a particular task, the Power Tower, which we have found to provide several opportunities to engender and draw attention to quantitative reasoning. We designed several aspects of the Power Tower activity in response to the PSTs' confusion during Interview Task 2. In addition to implementing this task during the eleventh session of the teaching experiment, we have used the task numerous times with students, in-service teachers, and preservice teachers. For that reason, what follows is an outline of how the task typically proceeds, with references to specific events that occurred during the teaching experiment. We also provide an example outcome drawn from the teaching experiment.

The Power Tower Activity

The Power Tower activity begins with the PSTs watching the video[5] of an amusement park ride featured at Cedar Point in Sandusky, Ohio. The video portrays riders on the Power Tower, a ride in which individuals are seated and secured in a car facing out on a tall vertical tower (Figure 6). The video shows the ride launching the riders from the bottom of the tower to the top before falling back toward the ground. The riders then bounce up and down, similar to a bungee jumper, before returning to the ground.

Figure 6. **The Power Tower.**

Implementation Overview

After playing the video and asking the PSTs to describe as many quantities in the situation as possible (e.g., rider's distance from the ground, rider's distance from the top, vertical speed, time elapsed since the rider was launched), we break the PSTs into groups of 3–4 and pose the following: "Sketch (we are not looking for absolute accuracy) a graph that relates the *vertical distance* from the ground of an individual and the individual's *total distance* traveled (assume their feet were touching the ground at the beginning of the video)." With the video on loop, we pass out worksheets with labeled axes on which to sketch their graphs. Half of the groups receive a prompt with total distance on the horizontal axis and vertical distance on the vertical axis, while the other groups have total distance on the vertical axis and vertical distance on the horizontal axis. The PSTs are not aware of the orientation discrepancy. After the groups are given sufficient time to work on the task, all sketches are hung around the room and the instructor opens the floor to a class discussion with the goal of comparing and contrasting the solutions.

5 See **http://www.youtube.com/watch?v=HrGpGMrkUrM** for a video with two versions of the ride.

Activity Design

We designed the task with many considerations. First, in attempting to graph distance from the ground versus total distance traveled, PSTs need to carefully consider how the two quantities covary. When considering the quantities presented in the situation, PSTs need to discover that total distance and vertical distance covary at a constant rate. Figure 7 presents such a relationship, composed of line segments such that during any up or down portion of the trip, both quantities change by equivalent amounts in magnitude. Although many groups arrive at such (correct) graphs by the end of their group work, several groups first draw graphs similar to Figure 8.

When drawing curved graphs like those in Figure 8, it is typical for PSTs to explain that the graph is curved because the riders are “speeding up” and “slowing down.” In fact, each small group of PSTs working on the problem during the teaching experiment initially drew curved graphs and gave such an explanation. Such an outcome provides an opportunity to direct the PSTs to more deeply consider the quantities of the situation. One manner in which to do this is to ask, “Tell me what quantities you are talking about when you say ‘speeding up’ or ‘slowing down’?” During the teaching experiment, asking this question led to the PSTs identifying that speed involves a particular form of rate of change that relates changes in distance and time. They then noted that because the graph involves two distances, the two distances must be the quantities that are compared in order to determine the appropriate rate of change relative to the graph.

In addition to raising important aspects of rate of change, we use this task to discuss other aspects of function. This is accomplished by raising prompts like those in Interview Task 2 either during group work or during the whole-class discussion. For instance, during the teaching experiment the PSTs who produced the graph in Figure 7a expressed discomfort with their graph and questioned whether or not the graph represents a function because it did not pass the vertical line test. As the PSTs considered graphs of different orientations, this discomfort eventually led to the interaction and conclusions presented in the following section.

Activity Outcome Example

Excerpt 3 presents a discussion that followed a 15-minute class debate leading to the PSTs settling on the graphs in Figure 7. At this time, the instructor decided to transition the discussion to considering the different axis orientations. Standing in contrast to Excerpt 2, the PSTs, including Monica, maintained a focus on the quantities of the situation when discussing “function.”

A partial explanation for the PSTs’ focus on quantities in this interaction is that they experienced a previous activity involving graphing trigonometric functions in different axis orientations. Still, it is important to highlight their focus on quantities when discussing function as a unique mapping, as during the previous activities we had not drawn their attention explicitly to this aspect. Thus, the activity gave the PSTs a repeated opportunity to reason about quantitative relationships in the context of function and reach the sensible conclusion that the graph—meaning a relationship between two quantities—is and is not a function all at once.

In our experience, students and teachers do not always reach the conclusion shown in Excerpt 3 so fluidly. One way in which we have generated conversations with other students and teachers like the one shown in Excerpt 3 is to draw their attention to the seemingly contradictory claim of both graphs representing the same relationship, yet only one graph representing a function using the verti-

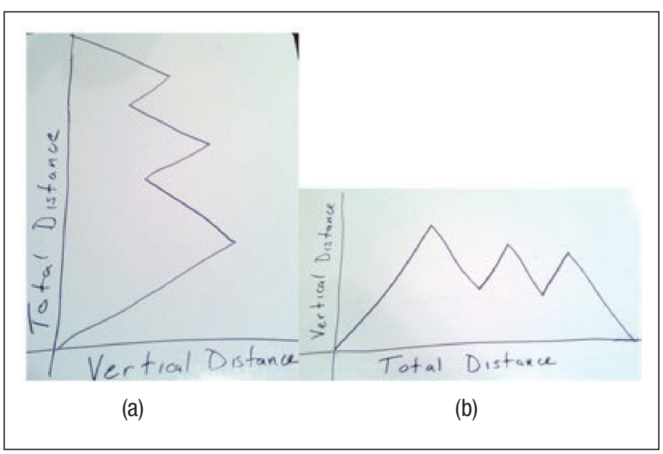

Figure 7. **Correct Power Tower sketches.**

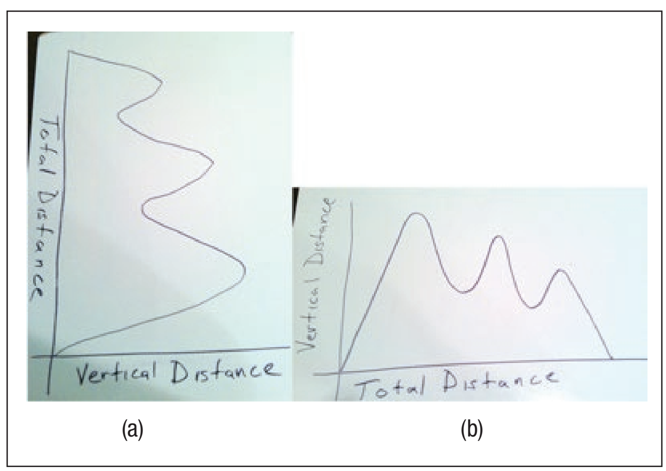

Figure 8. **Power Tower sketches that incorporate speed.**

Excerpt 3

Condensed discussion of the Power Tower activity and function[6]

Instructor: There's another observation I have. Those two graphs look different.

Rachel: You gave us different papers! [*class laughs*]

Monica: That was sneaky. [*Chatter around the classroom*]

Instructor: So what do you guys think about that?

Janice: It's the same relationship, it's just different . . .

Cassie: Like what we talked about with the sine and cosine, how we looked at the inputs and outputs differently. It's just the same thing with this graph.

Instructor: OK, so we've hit on something here. I heard Janice say [*referring to an earlier comment*], she brought up the idea of a function. So there's two simplified graphs. [*Directs students' attention to projection of the graphs in Figure 7*] Functions? Not functions?

Janice: Depends on what you call your input and what you call your output.

Instructor: So say a little bit more.

Cassie: So like, where, this graph that's sideways [*Figure 7a*]. . . . It's the same graph as this one [*Figure 7b*], just the sides are labeled different. The sides are like representing different quantities. So like the vertical axis for the first graph is the total distance traveled whereas the horizontal axis in the second graph is the total distance traveled.

Instructor: OK. So . . . for the graph on the left [*Figure 7a*], function or not a function?

Ross: Well, it depends which axis is your input and which axis is your output.

Instructor: So say a little bit more, Ross.

Ross: OK, the graph on the left, if your inputs were actually your vertical axis, the outputs were your horizontal axis, then that would be a function cause you would have a, let me do this right, a unique output for every input.

Instructor: So what do you guys think?

Students: [*General affirmation through shaking heads*] Yeah.

Instructor: OK, but then you're saying could we look at that graph and say it's not a function? [*class affirms*] How so?

Ross: If the x-axis is your input and y-axis is your output.

Janice: Cause if x is your input and then y is your output then you're going to have 1, 2, 3, 4 ,fi- like 5 points for an input. And that's [*Figure 7a*] not a function.

Instructor: OK, the second graph [*Figure 7b*], is it a function or is it not a function?

Janice: It is if you think about your x axis being your input.

Instructor: By x-axis here we're meaning . . .

Students: The horizontal axis is the input.

Instructor: And the vertical?

Students: Your output.

Instructor: OK. Not a function if?

Students: The vertical axis is your input.

6 Because this discussion occurred over five minutes of real time, this transcript has been condensed to highlight critical comments.

cal line test. In the ensuing conversation, students and teachers have the opportunity to grapple with their function understandings while the instructor acts as a facilitator when necessary in order to draw their attention to the quantities of the situation. After the students and teachers determine that, regardless of graphical representation, vertical distance is a function of total distance while total distance is not a function of vertical distance, the instructor can draw their attention back to the vertical line test. In particular, the instructor can emphasize the vertical line test in terms of classifying relationships between quantities, rather than just reporting a number of intersections. Such conversations allow students and teachers to consider the fact that both graphs can be conceived as conveying a particular type of relationship in a way that seems both natural and sensible.

Supporting PSTs' Quantitative Reasoning

Since many teachers and PSTs have not had substantive quantitative reasoning experiences, it is important that their professional development opportunities provide them repeated experiences to engage in such thinking. Consistent with previous work in the area (Thompson, 1993; Thompson, Philipp, Thompson, & Boyd, 1994), the above tasks and examples illustrate a few general strategies for creating such experiences. Each of these strategies—breaking conventions, using quantitatively rich and open situations, and keeping a focus on quantities and relationships – can be used as design and implementation principles for mathematics teacher educators interested in supporting preservice or in-service teachers' development of quantitative reasoning.

Breaking conventions: Each of the three tasks described above broke conventions commonly found in school mathematics (e.g., input or *x* on the horizontal axis). Implementing noncanonical tasks can serve two purposes. Although in many cases it is appropriate to use typical tasks taken from middle and high school curricula, PSTs "know the answer" to many of these tasks and are less likely to legitimately engage with the task (Blanton, 2002). We suggest that teacher educators implement noncanonical tasks in order to create the perturbations and curiosity necessary to engage PSTs in novel reasoning, and particularly quantitative reasoning. A second purpose for breaking conventions is gaining insight into PSTs' quantitative reasoning by creating situations that are difficult to approach with knowledge not based in quantitative reasoning (e.g., Excerpt 1). By introducing such situations, educators can identify instances when PSTs' responses are tied to procedural understandings versus those who focus on quantities and their relationships and thus who are more likely to make instructional decisions based on this knowledge.

Using quantitatively rich situations: When attempting to promote quantitative reasoning, it is critical to use tasks in which PSTs are required to consider numerous quantities and make distinctions between these quantities. The Power Tower activity represents one such task that can be deemed *quantitatively rich*, meaning that it is a situation composed of numerous quantities that must be clearly distinguished and related to solve the problem (Thompson, 1993). By design, students' activity during this task can raise issues of rate of change that illustrate the importance of focusing on the relevant quantities (e.g., two distances as opposed to speed).

It is important to note that we do not intend *quantitatively rich* to be interpreted as only involving so-called real-world or applied problems. A problem need not consist of an applied context to be quantitatively rich. For instance, Interview Task 1 was not set within an applied context. Yet the task requires reasoning about rate of change as a relationship between the values *x* and *y*. In short, a task is not quantitatively rich in and of itself. Instead, a task is made quantitatively rich by educators supporting particular engagement with the task. As mentioned above, one way to support such engagement is to design tasks that break conventions.

Keeping a focus on quantities and their relationships: Because a task is not inherently quantitatively rich, but instead becomes quantitatively rich through PSTs' engagement with the task, it is important that PSTs develop a disposition toward quantitative reasoning. This can be accomplished by maintaining a focus on quantities and their relationships—across various topics, tasks, and content—when interacting with PSTs. One way to do this involves avoiding the use of pronouns and other vague referents. When PSTs use words like "it," "that," or "this," ask them to restate their thoughts by explicitly referencing the quantities of the situation. Similarly, when PSTs use various mathematical referents (e.g., "slope," "the graph," or "the intercept"), ask them to explain their claims in terms of quantities and their relationships. For instance, when PSTs use the term *slope* in tasks that break conventions, prompt them to describe what they mean by slope and draw out the importance of discussing slope as the rate of change between two quantities. A specific suggestion for drawing out this importance is posing several graphs like those in Figures 3 and 4 and asking the PSTs: As we move from graph to graph, does the slope change? This conversation can highlight distinctions between different meanings for slope, including slope as perceptual tilt and slope as rate of change.

We also suggest that teacher educators implement tasks in ways such that quantitative reasoning is both an integrated and important part of PSTs' classroom experience. During the Power Tower activity, we allow the PSTs to spend 30–45 minutes sketching an appropriate graph, and then another 15–20 minutes considering the collection of presented graphs. This time enables the PSTs to construct a well-developed understanding of the quantities of the situation, which then supports discussions like that in Excerpt 3. Without giving PSTs such time to develop a robust understanding of the quantities and their relationships, a conversation like that in Excerpt 3 is unlikely. For this reason, when implementing tasks that focus on quantitative reasoning we suggest that educators remain flexible in the amount of time they provide PSTs to work on and discuss tasks. In our work, we allow PSTs to engage in tasks for as long as necessary provided that they are maintaining substantive discussions that involve quantitative reasoning. Providing PSTs ample time to consider and explore relationships between quantities helps them develop personal experiences with how important such reasoning is in the learning of mathematics.

Concluding Thought

In this article, we argue for the importance of quantitative reasoning in school mathematics and present evidence of the issues that arise when topics like function are learned in a way that mistakes conventions for mathematical facts. As noted, a benefit of quantitative reasoning is that it enables exploring mathematical ideas in noncanonical representations (e.g., input on the vertical axis) and in a variety of settings (e.g., polar coordinate systems). But if K–16 students (including PSTs) are not given the opportunity to experience noncanonical representations and a variety of settings, then they are not given the opportunity to experience quantitative reasoning in such a way. For this reason, we reiterate the potential benefits of implementing tasks that present noncanonical situations to students and teachers. Not only should educators use these tasks to perturb students' and teachers' thinking in ways that engender quantitative reasoning, but educators should also use such tasks to develop more flexible understandings of key secondary concepts (e.g., function and rate of change).

References

Blanton, M. L. (2002). Using an undergraduate geometry course to challenge preservice teachers' notions of discourse. *Journal of Mathematics Teacher Education, 5*, 117–152.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247–285.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45–75.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Clark, P. G., Moore, K. C., & Carlson, M. P. (2008). Documenting the emergence of "speaking with meaning" as a sociomathematical norm in professional learning community discourse. *The Journal of Mathematical Behavior, 27*(24), 297–310.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13.

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Retrieved from http://www.corestandards.org/assets/CCSSI_Math Standards.pdf

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439–478.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1–64.

Lobato, J., & Bowers, J. (2000). *Three perspectives in research on functions: Multi-representational, quantitative, and phenomenological.* Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Lobato, J., Rhodehamel, B., & Hohensee, C. (2012). "Noticing" as an alternative transfer of learning process. *Journal of the Learning Sciences, 21*(3), 433–482.

Montiel, M., Vidakovic, D., & Kabael, T. (2008). Relationship between students' understanding of functions in Cartesian and polar coordinate systems. *Investigations in Mathematics Learning, 1*(2), 52–70.

Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 75–92). Laramie, WY: University of Wyoming.

Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics, 83*(2), 225–245.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior, 31*(1), 48–59.

Moore, K. C., & LaForest, K. R. (in press). Connecting students' trigonometry with quantitative reasoning. *Mathematics Teacher*.

Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior, 32*(3), 461–473.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27–42). Washington, DC: Mathematical Association of America.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND.

Smith III, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). New York, NY: Lawrence Erlbaum Associates.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillside, NJ: Erlbaum.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Calculational and conceptual orientations in teaching mathematics. In A. Coxford (Ed.), *1994 yearbook of the NCTM* (pp. 79–92). Reston, VA: NCTM.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165–208.

Thompson, P. W. (1994). Students, functions, and the undergraduate mathematics curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education, I* (Vol. 4, pp. 21–44). Providence, RI: American Mathematical Society.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, DC: Falmer Press.

Authors' Note

The research reported in this article was supported by the National Science Foundation under grant number EHR-0412537 (Principal Investigator Marilyn Carlson). Thank you to Dr. Pat Thompson, not only for his foundational work on quantitative reasoning, but also because the Power Tower activity is a modified version of a task that we observed him using with in-service teachers. Thank you to Dave Liss and Stacy Musgrave for their many contributions to this project. All opinions expressed are solely those of the authors.

Correspondence concerning this manuscript should be addressed to Kevin C. Moore, 105 Aderhold Hall, Department of Mathematics and Science Education, University of Georgia, Athens, GA 30602. E-mail: **kvcmoore@uga.edu**.

Authors

Kevin C. Moore, 105 Aderhold Hall, Department of Mathematics and Science Education, University of Georgia, Athens, GA 30602; kvcmoore@uga.edu

Jason Silverman, 213 Korman Center, Mathematics Learning and Teaching Program, School of Education, Drexel University, Philadelphia, PA 19104; silverman@drexel.edu

Teo Paoletti, 105 Aderhold Hall, Department of Mathematics and Science Education, University of Georgia, Athens, GA 30602; paolett2@uga.edu

Kevin R. LaForest, 110A Aderhold Hall, Department of Mathematics and Science Education, University of Georgia, Athens, GA 30602; laforesk@uga.edu

Appendix

Session Summary and Task Examples

During the 15 class sessions that comprised the teaching experiment, we implemented numerous tasks designed to support the PSTs' quantitative reasoning. The first five sessions used tasks that are part of a research-based precalculus curriculum project, *Pathways to Calculus: A Problem Solving Approach* (Carlson & Oehrtman, 2010), to introduce the PSTs to quantitative and covariational reasoning in the context of trigonometric functions. Because these tasks are a part of the *Pathways* curriculum and reported on elsewhere (Moore, 2012, 2013, in press), we direct the reader to contact the *Pathways* author team (**https://rationalreasoning.net**) for information regarding the tasks and curriculum.

Using a Second Coordinate System

Following the trigonometry activities, five sessions explored the polar coordinate system in order to use a context in which the graphical conventions were different from those of the Cartesian coordinate system. We conjectured that different graphical conventions would challenge PSTs who held understandings tied to particular conventions of the Cartesian coordinate system in ways that simultaneously shaped their understandings of graphing relationships in the Cartesian and polar coordinate systems. We direct the reader to Moore, Paoletti, & Musgrave (2013) for a comprehensive overview of the sessions instructional tasks.

A majority of the tasks either: (1) presented a graph in the polar coordinate system and asked the PSTs to determine a formula for the graph and the corresponding Cartesian coordinate system graph or (2) tasked the PSTs with creating the polar coordinate system graph for a given formula. In the latter case, we also prompted the PSTs to compare their polar graphs to those they would have created in the Cartesian coordinate system. Regardless of task, our questioning focused on their explaining relationships between two quantities as opposed to describing global or perceptual characteristics like the shapes of the graphs (e.g., a spiral or rose). An example task is provided in Figure A1.

Based on the clinical interview outcomes like those from Interview Task 1, we used this particular task to problematize understandings of linear functions tied to visual cues; linear functions do not appear as a line in the polar coordinate system. This task gave the PSTs an opportunity to revisit graphing linear functions in the Cartesian coordinate system in order to compare such graphs to those they were producing in the polar coordinate system.

Illustrating our intention that the PSTs focus on describing the graphs in terms of covarying quantities, Kate gave the following explanation while moving between her two graphs to show that both graphs represented the same relationship (Excerpt A1).

Changing Axes Orientations

Stemming from the outcomes of Interview Task 1 and Interview Task 2, the last five teaching sessions focused entirely on graphing with multiple axes orientations in order to focus on the choice of labeling axes as arbitrary. That is, we intended for the PSTs to come to understand that a particular relationship can be graphed using different axes orientations without changing the underlying relationship. To accomplish this goal, we focused on two types of tasks: (1) tasks that challenged the convention of input on the horizontal axis and/or (2) tasks that challenged the convention of how numbers are orientated on the axes.

The Power Tower activity represented one such task. Another set of tasks explored graphing various function classes (e.g., linear, quadratic, and exponential) in

Consider a function $r(\theta) = 2\theta + 1$.

a. Graph the relationship in the polar plane.
b. What do values 2 and 1 represent in terms of the polar plane?
c. How does your graph in part a compare to the graph of $y = 2x + 1$ in the Cartesian coordinate system? What do the values 2 and 1 represent in terms of the Cartesian coordinate system?

Figure A1. **A polar coordinates task with a linear function.**

Excerpt A1

Kate working with a linear function as reported in Moore, Paoletti, and Musgrave (2013, p. 466)

Kate: They went out by two, like you know here [*pointing at the two in the formula* $r = 2\theta + 1$] the slope is like two [*tapping along the Cartesian graph*].

Int.: This has no slope [*pointing to the polar graph*]. . .

Kate: No, I'm relating the slope here [*pointing to the Cartesian graph*] to the difference in the radius of two each time *[tapping along the polar graph]. Like [the radius is] one, three, five, seven, nine, eleven [pointing to the corresponding points on the polar graph]; [the radius] increases by two.*

Later in the interaction

Kate: That's cool . . . because you'd never see this [*referring to the graph of* $r = 2\theta + 1$ *in the polar plane*] and be like, that's a linear function.

a variety of axes orientations. For instance, we gave the PSTs axes oriented like those in Figure A2 and asked them to graph the same function on each set of axes.

Suggesting that the combination of graphing in the polar coordinate system and considering axes orientations during the Power Tower activity influenced PSTs' focus on quantitative reasoning, we began observing significant shifts in the PSTs' activity at this point in the teaching sessions. While a majority of the PSTs encountered difficulty making sense of the student's solution in Interview Task 1 during the interviews at the beginning of the semester, we did not observe PSTs relying on visual cues when graphing in various orientations during the teaching sessions. Instead, the PSTs predominantly considered how to represent covarying quantities relative to the provided orientations. In fact, several PSTs asked the instructor why their previous classes had not focused on reasoning about quantities and their relationships under such conditions. The PSTs explained that they found value in such reasoning because it provided a coherent way to approach graphing relationships regardless of coordinate system or orientation.

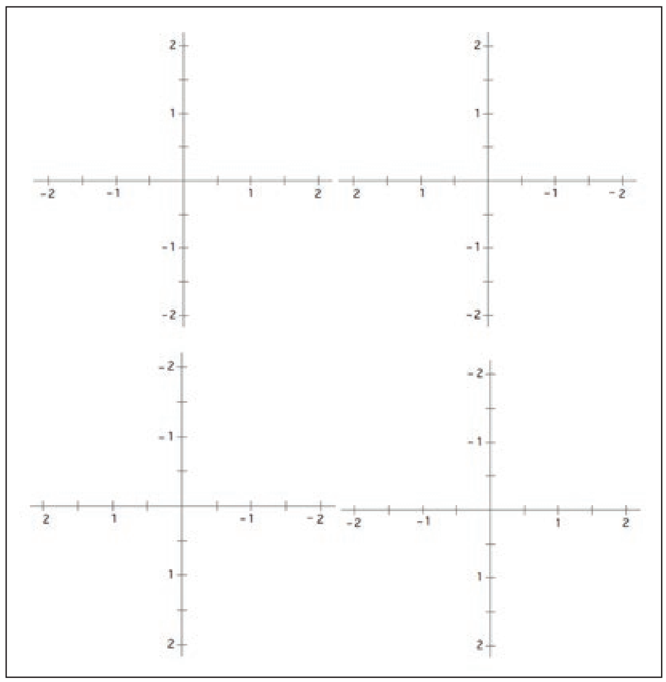

Figure A2. **Four axes orientations for graphing in the Cartesian coordinate system.**

Culminating Task

On the culminating task (see Figure A3), which we designed to offer insights into what the PSTs had interpreted to be *quantitative reasoning*, we asked the PSTs to choose a course topic and present an illustration of quantitative reasoning with respect to that topic. As one illustrative example that suggests the PSTs had developed understandings compatible with those we intended, a group of PSTs presented several graphs (Figure A4) to illustrate how graphs using different orientations can each convey the same quantitative relationship: the cosine function in this

Choose a course topic and present an illustration of quantitative reasoning with respect to that topic. Your narrative should provide an example task and solutions to that task such that your solutions clearly illustrate quantitative reasoning.

Also, prepare a presentation for the class that clearly conveys the role of quantitative reasoning in your topic. Be creative and feel free to have the class participate as if they are your students.

Figure A3. **Culminating task for the PSTs' content course.**

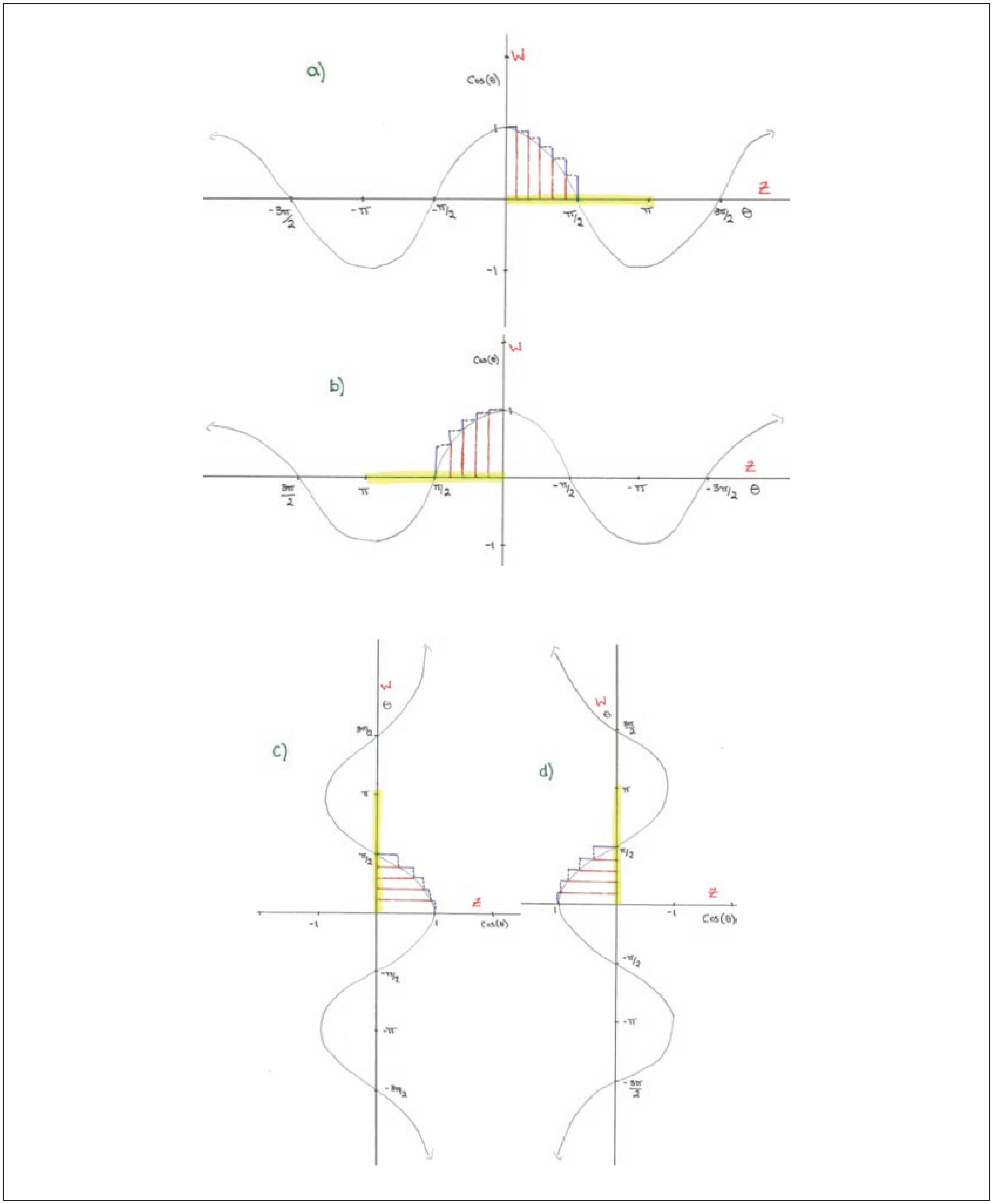

Figure A4. **Four graphs, each illustrating the cosine function.**

case. Collectively, the PSTs provided descriptions of how quantitative reasoning related to a particular course topic, with most PSTs avoiding the use of pronouns and instead choosing to discuss tasks and topics in terms of quantities and their relationships.

References

Carlson, M. P., & Oehrtman, M. (2010). *Pathways to calculus: A problem solving approach*. Rational Reasoning.

Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 75–92). Laramie, WY: University of Wyoming.

Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics*, 83(2), 225–245. doi: 10.1007/s10649-012-9450-6

Moore, K. C. (in press). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*.

Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior*, 32(3), 461–473.

(Return to page 148)