Quantitative Reasoning and the Sine Function: The Case of Zac.

Kevin C. Moore
University of Georgia

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138.

Available at: http://www.jstor.org/stable/10.5951/jresematheduc.45.1.0102

Journal for Research in Mathematics Education
2014, Vol. 45, No. 1, 102–138

Quantitative Reasoning and the Sine Function: The Case of Zac

Kevin C. Moore
University of Georgia

A growing body of literature has identified quantitative and covariational reasoning as critical for secondary and undergraduate student learning, particularly for topics that require students to make sense of relationships between quantities. The present study extends this body of literature by characterizing an undergraduate precalculus student's progress during a teaching experiment exploring angle measure and trigonometric functions. I illustrate that connecting angle measure to measuring arcs and conceiving the radius as a unit of measure can engender trigonometric meanings that encompass both unit circle and right triangle trigonometry contexts. The student's progress during the teaching experiment also indicates that a covariation meaning for the sine function supports using the sine function to represent emergent relationships between quantities in novel situations.

Key words: Angle measure; Covariational reasoning; Precalculus; Quantitative reasoning; Trigonometry

The historical development of trigonometric functions extends over a period of two-millennia (Bressoud, 2010; Van Brummelen, 2009), with many advances occurring within triangle and circle contexts. Circle trigonometry partly emerged from the Greeks' study of the heavens and remained the predominant focus for nearly two thousand years, while triangle trigonometry gained standing during the 16th century and has since grown in use. Just as the use of trigonometric functions has evolved within two predominant contexts, the treatment of trigonometric functions in U.S. curricula has emphasized both right triangle and circle contexts, with the foregrounded or initial context depending on the curriculum and course. Precalculus books may introduce trigonometric functions in the context of the unit circle (e.g., Axler, 2009; Stewart, Redline, & Watson, 2012), whereas a geometry course following the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) is likely to introduce and explore trigonometric ratios in a right triangle context without necessarily introducing the formal function names.

The research reported in this paper was supported by the National Science Foundation under grant number EHR-0412537. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation. Thank you to Dr. Andrew Izsák for critiquing previous versions of this article. Also, the current article is based on dissertation research that was completed under the guidance of Dr. Marilyn P. Carlson, and I thank her for numerous contributions to the study.

Due to the emphasis in both circle and triangle contexts, it is important that students construct meanings for trigonometric functions that are applicable to both contexts. Unfortunately, researchers have illustrated that students and teachers are not developing such meanings (Brown, 2005; Fi, 2003, 2006; P. W. Thompson, Carlson, & Silverman, 2007; K. Weber, 2005), and several researchers have partly attributed this problem to students' and teachers' restrictive understandings of topics foundational to trigonometric functions, such as angle measure (Akkoc, 2008; Topçu, Kertil, Akkoç, Yilmaz, & Önder, 2006). P. W. Thompson (2008) argued that greater attention should be paid to identifying ways of thinking that are beneficial for students' learning both in the moment and longitudinally. He called for leveraging foundational topics, like angle measure, in ways that foster compatible meanings for trigonometric functions among their various contexts. He also asserted that predominant approaches to trigonometry, particularly common treatments of angle measure, create an unnecessary divide between circle and triangle contexts, a divide that is reflected in the body of literature on the teaching and learning of trigonometry (e.g., Akkoc, 2008; Moore, 2012, 2013).

Review of Literature on Trigonometry and Angle Measure

Although research on the teaching and learning of trigonometry is sparse (K. Weber, 2005), available literature conveys several themes. First, researchers have argued that underdeveloped angle measure understandings contribute to teachers' difficulties with trigonometric functions (e.g., Akkoc, 2008; Fi, 2003; Topçu et al., 2006). Akkoç (2008), Topçu, Kertil, Akkoç, Yilmaz, and Önder (2006), and Fi (2003) each characterized preservice and inservice teachers' radian measure meanings as reliant on degree measures because many teachers were constrained to describing radian measures in terms of degree measures. In fact, not one preservice teacher in Fi's study defined radian measure as a multiplicative relationship between an arc and a circle's radius. Akkoc (2008) and Topçu et al. (2006) argued that an implication of preservice and inservice teachers' reliance on degree measures, in combination with the predominant use of degree measures in right triangle contexts, is that their meanings for trigonometric functions have a tendency to give primacy to or be restricted to right triangle contexts. As Akkoc (2008) illustrated, such meanings can result in a reluctance to consider input values to trigonometric functions other than degree measures, which becomes problematic when defining trigonometric functions in terms of radian measures.

A second theme that emerges from the body of literature on the teaching and learning of trigonometry focuses on mathematics students' capacity to leverage and move among various geometric objects and representational systems associated with trigonometric functions. Working with Honors Algebra 2/Trigonometry students, Brown (2005) investigated students' coordination of various representations and contexts typically found in the teaching of trigonometric functions. Brown found that students' meanings for trigonometric functions were often constrained to triangles and ratios, as opposed to circles and numbers not represented as ratios. She also illustrated that thinking about angles in terms of rotations can support students in connecting angles and the unit circle to graphs of trigonometric functions. Brown further emphasized the importance of students connecting coordinates to directed lengths when attempting to connect the unit circle to graphs of trigonometric functions. Echoing Brown's findings, Hertel and Cullen (2011) noted the importance of supporting preservice teachers in reasoning about directed lengths when exploring trigonometric functions.

K. Weber (2005) reported that undergraduate mathematics students encounter difficulty leveraging geometric objects to explain various properties of trigonometric functions. For instance, when approximating $\sin(\theta)$ for various values of θ, several students from a traditional trigonometry course stated that they needed an appropriately labeled triangle to complete the task. No student was able to successfully create such a triangle. K. Weber (2005) concluded, "What these students seemed to lack was the ability or inclination to mentally or physically construct geometric objects to help them deal with trigonometric situations" (p. 103). K. Weber also conducted an experimental course in trigonometry that was informed by Gray and Tall's (1994) theoretical notion of procept. The students in the experimental course improved their performance because of their ability to apply the unit circle to novel problems. Yet, K. Weber hesitated to claim that educators should expect all unit circle approaches to improve student understanding. Instead, he stressed that regardless of context, it is necessary that students construct the geometric objects of trigonometry as tools for reasoning.

Several other studies and dissertations have contributed to the literature base on trigonometric functions, particularly in relation to modeling and the use of technology (e.g., Blackett & Tall, 1991; Doerr, 1996; Steckroth, 2007; K. A. Thompson, 2007). Of relevance to the present study, Doerr (1996) characterized secondary students' application of trigonometric functions in a force, vector, and modeling setting. Like Brown (2005), she highlighted the importance of students' capacity to move among different representations (e.g., graphs, tables, and diagrams) when using trigonometric functions and suggested that using a modeling approach may support such representational fluency.

Collectively, the aforementioned research has provided insights into the teaching and learning of trigonometry that give direction for exploring particular meanings at play during a student's construction of trigonometric functions. For instance, modeling forms an area in which students face much difficulty. Thus, studying students' activity in terms of the mental actions that researchers have identified as critical for modeling (cf. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) may provide deeper insights into how to incorporate Doerr's suggestions. Additionally, researchers have repeatedly identified individual's difficulties with angle measure and the geometric objects associated with trigonometric functions, indicating that much is to be learned about how to support and draw on student thinking in these areas.

Research Questions

The persistence and breadth of student and teacher difficulties in trigonometry calls for focused efforts that identify ways of developing meanings[1] for trigonometric functions in both unit circle and right triangle contexts. By building on the aforementioned researchers' suggestions (e.g., Brown, 2005; Doerr, 1996; P. W. Thompson et al., 2007) and focusing tightly on one student's mathematical thinking, I examine how reasoning about quantities and relationships between quantities may support a student in developing meanings applicable to both contexts. I also pay heed to the suggestion that students' angle measure meanings must be more carefully considered in the teaching and learning of trigonometry (Akkoc, 2008; Fi, 2003; Steckroth, 2007; P. W. Thompson, 2008; Topçu et al., 2006). I put this suggestion into action by explaining how a student's angle measure meanings influenced his construction of the sine function. In doing so, I illustrate how angle measure meanings rooted in quantitative reasoning can provide a foundational way of thinking for the development of the sine function. The research questions guiding the study are:

RQ1: What meanings of the sine function does a student develop during an instructional sequence emphasizing quantitative and covariational reasoning?

RQ2: How do a student's angle measure meanings, including the radius as a unit of measure, influence his sine function meanings?

Theoretical Framing

Quantitative reasoning (P. W. Thompson, 1990) and covariational reasoning (Carlson et al., 2002) characterize the mental activity of making sense of quantities and relationships between quantities. A growing body of literature (e.g., Castillo-Garsow, 2010, 2012; Confrey & Smith, 1995; Ellis, 2007; Moore, 2012; Oehrtman, Carlson, & Thompson, 2008; Rasmussen, 2001; P. W. Thompson, 1994b, 2011) has identified such reasoning to be critical in supporting student learning in secondary and undergraduate mathematics, particularly in the context of function. Yet, as outlined above, students and teachers often develop meanings for trigonometric functions that are not built on relationships between covarying quantities. For instance, individuals may remember the acronym SOHCAHTOA (Sine is Opposite over Hypotenuse, Cosine is Adjacent over Hypotenuse, Tangent is Opposite over Adjacent), but those same individuals are likely to encounter obstacles when attempting to connect the ratios implied by this acronym to values associated with the unit circle or graphs of the trigonometric functions (Brown, 2005; K. Weber, 2005). Contributing to these difficulties, it does not appear that students or teachers quantify angle measure—a quantity—in a way that supports understanding trigonometric functions in terms of relationships between quantities (Akkoc, 2008; Brown, 2005; Moore, 2012, 2013; P. W. Thompson, 2008).

[1] I use the term *meanings* to refer to the mental actions and ways of operating to which an individual assimilates a particular situation; to imbue a situation with a meaning (or understanding) is to assimilate the situation to a scheme (Skemp, 1979; P. W. Thompson, 2013). Because assimilation is the source of meanings, individuals construct meanings by repeatedly engaging in particular mental actions and reflecting on those experiences (Piaget, 1977/2001).

It is not entirely surprising that quantitative and covariational reasoning are not central to students' and teachers' trigonometric meanings. Researchers have repeatedly and emphatically characterized U.S. school mathematics as failing to develop meanings that stem from such reasoning (e.g., Cai & Wang, 2010; Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008; Smith & Thompson, 2007; P. W. Thompson, 2008, 2011, 2013). As P. W. Thompson (2008) wrote, "The lack of attention to meaning, I believe, is at the root of many problems that become visible only later in students' learning" (p. 33). P. W. Thompson used angle measure to illustrate how a lack of attention to meaning in mathematics education can be reconciled through an approach based in quantitative reasoning. Building on P. W. Thompson's (2008) discussion of angle measure, I examined the role of quantitative reasoning in students' quantification of angle measure (see Moore, 2012, 2013). In the following two sections, I extend the discussion provided in these previous works by outlining a system of trigonometric meanings that hinge on covariational and quantitative reasoning.

Covariational Reasoning and the Sine Function

Covariational reasoning, defined as the "cognitive activities [of an individual] involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354), is an important way of reasoning mathematically at the secondary and undergraduate levels. But, as Carlson and colleagues identified (Carlson, 1998; Carlson et al., 2002), even high-performing calculus students frequently encounter difficulty engaging in the mental actions necessary to describe and represent how quantities in a situation are changing in tandem. In light of their findings, the authors argued that an increased emphasis on supporting meanings that emerge from covariational reasoning is needed.

In order to characterize students' covariational reasoning abilities, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) developed a framework of five mental actions. Figure 1 illustrates their framework, with the mental actions also described in terms of the sine function. The mental actions are not to be interpreted as hierarchical. A student engaging in rate of change reasoning (MA5) does not imply that his reasoning is more sophisticated than a student engaging in amounts of change reasoning (MA3), nor does it imply that MA5 reasoning entails MA3 reasoning. In fact, Carlson (1998) identified students verbalizing a decreasing rate of change while sketching a smooth, concave down graph; but when explaining their meaning for a decreasing rate of change, some students were unable to unpack their rate of change descriptions in terms of amounts of change between the two quantities.

To further illustrate connections between covariational reasoning and the sine function, consider a task (Figure 2) that involves modeling circular motion by representing the covariational relationship between an angle measure and a directed length. By first considering an increasing angle measure over the first quarter of a revolution, one can identify that the (directed) vertical distance varies

Mental actions	Description of mental actions	Verbal behaviors related to the sine function
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other.	Verbalizing that the output sin(θ) varies as the angle measure, θ, varies.
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable.	Verbalizing an awareness of the *increasing* output value sin(θ) with an *increasing* value of angle measure θ (θ between 0 and $\pi/2$ radians).
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable.	Verbalizing that for successive increases in angle measure from 0 to $\pi/2$ radians, the output value sin(θ) increases and the *amount of increase* decreases.
Mental Action 4 (MA4)	Coordinating the average rate of change of the function with uniform increments of change in the input variable.	Verbalizing that the average rate of change of the output value sin(θ) with respect to angle measure θ *decreases* for successive uniform increases of angle measure θ between 0 and $\pi/2$ radians.
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function.	Verbalizing an awareness that the instantaneous rate of change of the output value sin(θ) with respect to angle measure θ *decreases* over the domain of 0 to $\pi/2$ radians.

Figure 1. The covariation framework and the sine function.

> Imagine a bug sitting on the end of a blade of a fan as the blade revolves in a counter-clockwise direction. The bug is exactly 3.1 feet from the center of the fan and is at the 3:00 position as the blade begins to turn. Create a graph that shows how the bug's vertical distance above the 9:00 to 3:00 diameter line varies with the angle swept out by the bug as it travels around the fan.

Figure 2. The fan problem and the sine function.

(MA1) and, specifically, that the vertical distance increases (MA2). Next, when considering equal changes of angle measure over this interval, the magnitude of increase in the vertical distance decreases (MA3)—see Figure 3. One can also conclude that the average rate of change of the vertical distance with respect to angle measure decreases over each successive equal increase of angle measure (MA4). Finally, imagining the rotation of a bug in a continuous manner can lead one to conclude that the vertical distance increases at a decreasing rate with respect

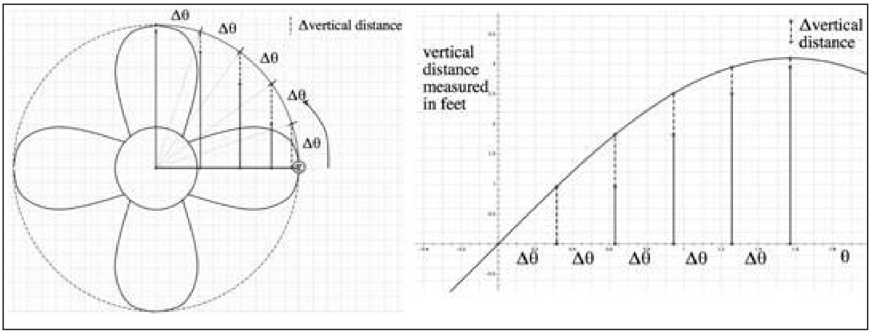

Figure 3. Representing amounts of change (MA3) in relation to the fan problem.

to angle measure (MA5). A similar line of reasoning can be used for the other three quarters of a revolution.

As highlighted by researchers (Carlson et al., 2002; Castillo-Garsow, 2012; Johnson, 2012a, 2012b; P. W. Thompson, 1994a, 1994b, 2013) and illustrated in the above example, no one mental action suffices in coming to understand how two quantities covary. The mental actions associated with Carlson and colleagues' framework can also entail different images of change (e.g., discrete MA3 versus continuous MA5) that reflect fundamentally different cognitive processes (Castillo-Garsow, Johnson, & Moore, in press), and much is left to be learned about how the coordination of different mental actions relates to students' understanding of particular topics such as trigonometric functions.

Quantitative Reasoning, the Sine Function, and Measuring in Radii

For an individual to engage in covariational reasoning, as defined by Carlson et al. (2002), it is presumed that the individual conceives of two quantities that covary. *Quantitative reasoning* (Smith & Thompson, 2007; P. W. Thompson, 1990, 2011) refers to the mental actions involved in conceiving of a situation (e.g., a person taking a Ferris wheel ride), coming to view the situation so that it entails attributes—called quantities—that admit a measurement process (e.g., the rider's angle of rotation and distance from the ground), and reasoning about relationships between these quantities (e.g., how the quantities vary in tandem). The quantitative structure that is produced by these mental actions is critical for the emergence of mathematical products (e.g., graphs and formulas) that, to the creator, each represent the same quantitative structure (Moore & Carlson, 2012; Moore, Paoletti, & Musgrave, 2013; Smith & Thompson, 2007).

A central premise to quantitative reasoning is that quantities and quantitative relationships are constructed over time (e.g., over the course of a problem, an instructional sequence, or even years) and in ways that are unique to the individual. For these reasons, quantities and their relationships should never be presumed (Izsák, 2003; Meira, 1995; P. W. Thompson, 2011). In the context of angle measure, an individual may conceive of angle measures as labels of geometric objects (e.g., a *line* is π radians and a *circle* is 2π radians) without an associated scheme for the structure of the measurement unit. On the other hand, an individual may come to understand angle measures in terms of a measurement process that defines a *multiplicative relationship*—a measure of one quantity being so many times as large as a second quantity—between a subtended arc and a circle's radius. Such a meaning enables a student to understand that a radian measure conveys an equivalence class of arcs that involves using the magnitude of the radius as the unit for multiplicative comparisons with arc lengths (Moore, 2013): For an angle to have a measure of 3 radians means that for any circle centered at the vertex of the angle, the angle subtends an arc length that is 3 *times as large* as that circle's radius.

In addition to forming a basis for radian measure, imposing the radius as a unit of measure can lead to a particular meaning for the sine function (and the unit circle) that encompasses all circles at once. Returning to the situation and graph in Figure 3, we choose to measure the directed vertical distance in radii rather than feet, and thus we obtain a circle and graph with values equivalent to the unit circle and the sine function, respectively (Figure 4). In the case of the sine function and in the context of a particular circle, the ordinate can be thought of as the measure of a directed vertical distance in radii. The abscissa represents a measure of an angle in radians, and this measure can be thought of as the measure of an arc in radii for a specific circle (Moore, 2013). More generally, the coordinate pair (π/4, sin(π/4)) conveys that for an arc length of π/4 radii counterclockwise from the standard position (3 o'clock) on any circle, the corresponding vertical distance above that circle's horizontal diameter is sin(π/4) times that circle's radius (or approximately 0.707 times as large as that circle's radius).

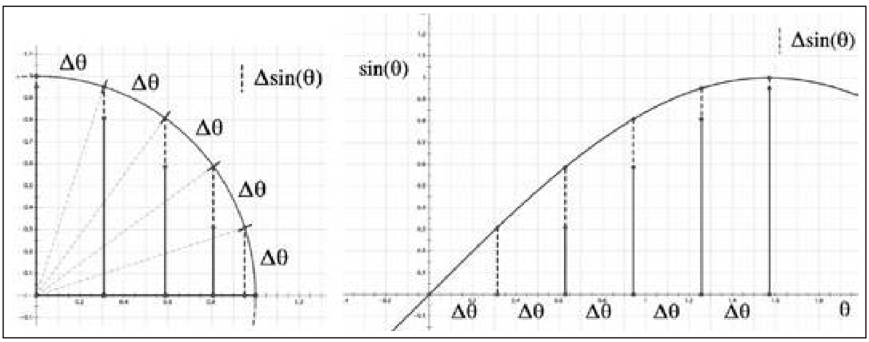

Figure 4. Measuring values relative to the radius and the sine function.

The choice to define the sine function using measures relative to the radius has several benefits. For one, the values convey numerical measures for every circle all at once because, regardless of the circle, one obtains equivalent numerical values when the quantities are measured relative to that circle's radius. These values are those typically conveyed on the unit circle (Moore, 2012; Moore, LaForest, & Kim, 2012). As others have noted (P. W. Thompson, 2008;

P. W. Thompson et al., 2007), another benefit of using both input and output values that stem from measuring in radii is that

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1,\text{ whereas }\lim_{x\to 0}\frac{\sin(x)}{x}\neq 1$$

if the numerical values of sin(x) and x do not stem from quantities measured in an equivalent unit magnitude. It follows that

$$\frac{d}{dx}\sin(x)=\cos(x)$$

when both the input and output values are those that stem from measuring in radii.

Developing meanings for angle measure and trigonometric functions that entail measuring arcs and lengths in a specified unit can also form important ways of reasoning for right triangle contexts. Figure 5 presents a right triangle with a labeled angle measure θ. If an individual holds arc meanings for angle measure, then he or she should conceive the labeled angle measure as conveying information about a family of arcs and circles centered at the vertex of the angle (Moore, 2013; P. W. Thompson, 2008). The hypotenuse of the right triangle forms the radius of one of these circles (Figure 6) and can thus be defined as the unit of measure for the formed arc length. Likewise, the hypotenuse (or radius) can be considered the unit of measure for each side of the right triangle. It follows that the sine function outputs the measure of the opposite leg relative to the hypotenuse length (e.g.,

$$\sin(\theta)=\frac{b}{h}\text{ or }h\sin(\theta)=b,\theta\text{ in radians).}$$

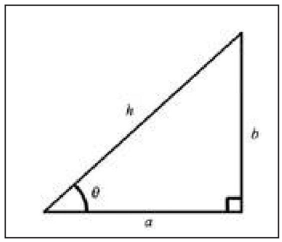

Figure 5. A right triangle.

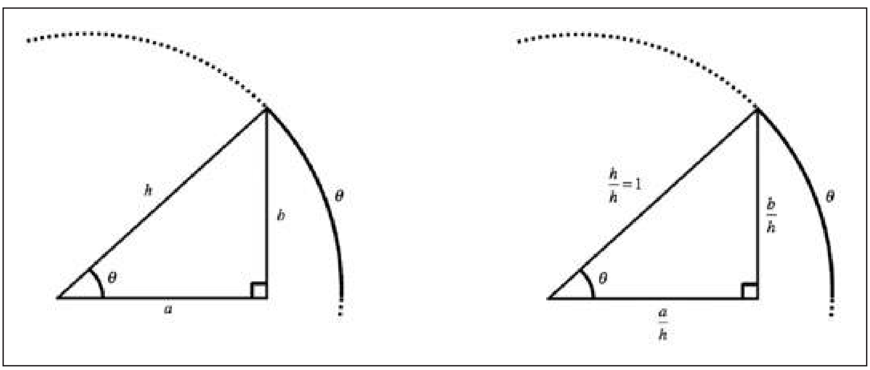

Figure 6. Using the hypotenuse as the radius of a circle (left) and as a unit (right).

Whether used in a circle or right triangle context, the sine function relates angle measures (with radians being the standardized unit) with lengths that are measured in a particular unit magnitude (the hypotenuse or radius). From this perspective, values typically associated with the unit circle and ratios typically associated with right triangles both emerge as multiplicative comparisons of quantities that hold across a class of objects (e.g., all circles or similar right triangles).[2]

The *conceptual analysis* (P. W. Thompson, 2008) provided in this section outlines a system of trigonometric meanings that centers on notions of covariation, angle measure, and measuring in radii. Although such an analysis does outline a system of meanings that may be beneficial for students to develop, the analysis does not address or provide an example of how the described ideas might emerge when working with students. In an attempt to better understand the development of such ideas, I conducted a teaching experiment with instructional design and data analysis efforts informed by the above conceptual analysis and literature base. In the following sections, I discuss the design and results of the teaching experiment, including how the proposed ideas informed the study.

Methodology

The present study is rooted in radical constructivist theories of learning (von Glasersfeld, 1995) and thus guided by the premise that each student's knowledge is fundamentally unknowable to any other individual. Reflecting this premise, a critical purpose of research is building models of *students' mathematics*, termed the *mathematics of students* (Steffe & Thompson, 2000). A *teaching experiment* foregrounds the act of conceptual analysis during the conduct of research and thus offers researchers a tool with which to continually generate, test, and modify hypothesized models of student thinking through continued interactions with students (Steffe & Thompson, 2000). These models are not to be interpreted as one-to-one representations of student thinking. Instead, the models are the researcher's best explanation—an explanation that is shaped by the researcher's own understandings, perspective, and ways of operating—of the meanings that might have contributed to the student's behavior.

Subject and Setting

This study is situated within a larger study that examined three undergraduate precalculus students' meanings for trigonometric functions and angle measure. All three students (one male, Zac, and two females, Judy and Amy) were enrolled in an undergraduate precalculus course at a large public university in the southwest United States. I chose the three students from a pool of volunteers based on the schedules of each volunteer. The three students were compensated for their time.

[2] I note that measurement plays a central role to this discussion. An extensive body of literature exists on this topic (see Steffe & Olive, 2010); however, a synthesis of this literature is beyond the scope of the present article.

This article focuses on Zac's meanings for the sine function; other aspects of the study are reported elsewhere (Moore, 2012), including a detailed examination of the students' angle measure meanings (Moore, 2013).

The precalculus class was involved in a design research project aimed at improving precalculus curriculum and teaching. Theory on processes of covariational reasoning and select literature about mathematical discourse and problem solving (Carlson & Bloom, 2005; Carlson et al., 2002; Clark, Moore, & Carlson, 2008) informed the design of the course. The present study commenced prior to the teaching of angle measure and trigonometric functions, which formed the last two course topics.

Zac was a full-time student at the time of data collection and received a "B" for his course grade. He was in his early 20s and an ethnomusicology/audio technology major. Zac had taken two high school algebra courses and two high school precalculus courses as recent as 3 years prior to the study. He had also taken a college calculus course 2 years prior to the study. The fact that Zac had taken precalculus and calculus courses prior to the study is not abnormal for students in an undergraduate precalculus course. In fact, another student involved in the study had also attempted a college calculus course prior to enrolling in the precalculus course. Such a situation may occur for a combination of reasons, including: (a) transfer from another institution, (b) change of major and requirements, (c) difficulty in calculus, and (d) a return from postponing his or her degree program. As is typical of precalculus courses, the course from which I selected Zac served a wide population of students with large variation in their backgrounds, and thus Zac is not an atypical student relative to the college precalculus landscape.

Data Collection and Analysis Methods

The teaching experiment consisted of five 75- to 120-minute sessions (Figure 7) that took place within a span of 15 days. The teaching experiment was limited to this period because of the need to minimize the time during which the students were absent from the precalculus course. The first four sessions included Zac, two peers,[3] an observer, and myself. The fifth session did not include any of Zac's peers in order to gain deeper insights into his thinking. I distinguish between the sessions that included Zac's peers and the session that did not by referring to the former as *teaching sessions* and to the latter as an *interview session*. I acted as the teacher–researcher for each session. Immediately after each session, I debriefed with the observer to discuss our observations during the sessions, form hypothesized models of the students' thinking, and design future sessions. I also conducted a preinterview with Zac. The preinterview followed the design of a clinical interview (Clement, 2000) and Goldin's (2000) principles of structured, task-based interviews. The preinterview occurred prior to the first teaching session and consisted of tasks that offered insights into Zac's angle

[3] One student's progress over the course of the study was compatible to Zac's progress. The third student's progress is reported in a manuscript that is in preparation.

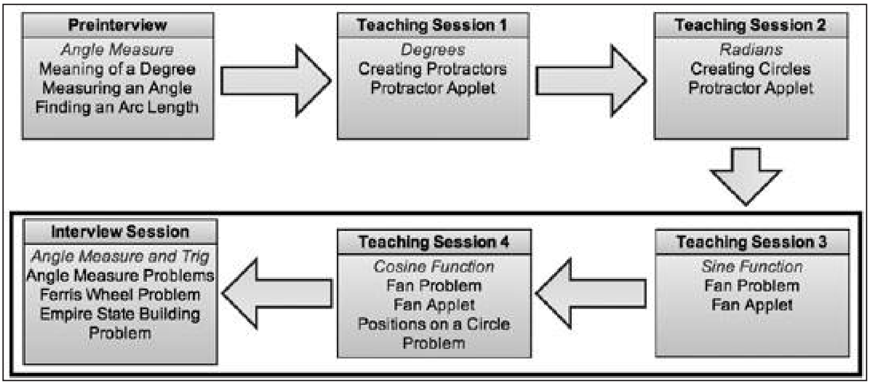

Figure 7. Progression of topics for the teaching and interview sessions with the boxed trio representing the extensive focus of this work.

measure meanings upon entering the study—see Moore (2013) for tasks and a detailed description of the preinterviews.

During the teaching sessions, the students typically began by working individually on the instructional tasks. As they worked, I observed and questioned them in order to develop models of their thinking. I also asked the students to share and discuss their thinking with one another to generate conversations that exposed various ways of thinking so that these ways of thinking became objects of their reflection. I facilitated the group discussions by posing questions and situations that I hoped would offer additional information about the students' thinking, particularly with respect to the ideas outlined in the previous section. Also, to support group discussion, the students and I worked on personal 2 ft × 3 ft whiteboards or from a classroom whiteboard. Because Zac was the only participant present during the interview session, he worked on tasks individually while I posed tasks and questioned him in order to test and refine my models of his thinking.

The data set included videos of all sessions, digitized researcher and observer notes taken during and between sessions, and digitized copies of student work. Two cameras captured each session, with one camera dedicated to student work and the other camera dedicated to teacher-researcher work and whiteboard work. The computer display used for dynamic computer applets was recorded. Student work was collected and digitized in order to confirm written work as inferred from the video data. Upon completion of each session, the camera and computer feeds were merged into one split-screen video file. Also, all digitized student work and written notes were organized chronologically and paired with appropriate video files in order to support retrospective analysis efforts.

Aligning with the teaching experiment methodology and the study that investigated Zac's angle measure meanings (Moore, 2013), the present study included ongoing and retrospective analysis elements. Ongoing efforts involved documenting all hypotheses made during the data collection. Upon completion of data collection, I used the video feeds and transcription software to transcribe all student and researcher utterances as well as all observable actions. Subsequently, I used an open and axial coding approach (Strauss & Corbin, 1998) to analyze the data set. In a first pass of the video and transcript data, I identified interaction sequences in which Zac's utterances and actions provided information about his mathematical thinking. With these instances identified, I utilized conceptual analysis techniques (Steffe & Thompson, 2000; P. W. Thompson, 2008) to characterize Zac's thinking. Whereas the conceptual analysis described above is of a theoretical nature, a methodological use of conceptual analysis is to develop and refine hypotheses of an individual's mental actions that explain the researcher's interpretations of the individual's observable behaviors (P. W. Thompson, 2008). After building initial models of Zac's mental actions, I analyzed the data set, which now included the hypothesized models, to identify contradictory and compatible models in order to refine and improve my models of his thinking. It was through this iterative process that I sought to build more viable models of his thinking and identify shifts in his thinking.

As an example of the data analysis process and to illustrate how the above description of trigonometric ideas informed the analysis efforts, a part of the analysis included comparing my models of his thinking to the mental actions detailed in Figure 1. Consistent with grounded theory (Strauss & Corbin, 1998), I did not assume that Zac would engage in covariational reasoning. Thus, I first developed my own explanations of his actions, and I then compared these explanations to the covariation mental actions in order to characterize his meanings in terms of the covariation framework. This process included providing alternative explanations for his activity that could not be captured by the framework. I then compared and contrasted my characterizations of Zac's covariational reasoning across the data set. Such a process enabled me to document shifts in his thinking and draw implications of such shifts (or lack thereof) relative to his meanings for the sine function and angle measure (see *RQ1* and *RQ2*).

Results

To provide background information on Zac's angle measure meanings, I first summarize the preinterview and the initial pair of teaching sessions reported elsewhere (Moore, 2013). I then discuss the remaining sessions of the teaching experiment in five phases that are organized chronologically around topical aspects of the teaching experiment:

1. Constructing a covariational relationship (Teaching Session 3)
2. Implications of using the radius as a unit of measure (Teaching Session 3)
3. Evaluating the sine function (Teaching Session 4)
4. Further investigating circular motion (Interview Session)
5. Extending to right triangles (Interview Session)

For each phase, I describe my models of Zac's reasoning, with attention given to how arc notions of angle measure, coordinating varying quantities (with reference to the mental actions in Figure 1), and conceiving the radius as a unit of measure enabled Zac to reason in compatible ways when tasked with problems involving circle or triangle contexts.

Angle Measure Meanings

Zac held restrictive angle measure meanings at the onset of the study, a finding that is compatible with previous characterizations of students and teachers (Akkoc, 2008; Topçu et al., 2006). During the preinterview, Zac conceived some angle measures as labels for geometric objects (e.g., a line is 180 degrees and a circle is 360 degrees). When given an angle measure that did not relate directly to a geometric object, like a line or a circle, he drew an angle with an openness that he estimated to have the given measure. Zac's activity during the preinterview also indicated that he had not quantified angle measure in terms of a process that involved systematically measuring and comparing attributes such as arcs and a circle's circumference. For instance, when given sufficient tools to estimate the measure of an angle (e.g., a compass, ruler, piece of string, and calculator), he explained that he did not know how to measure an angle with the available tools. He did mention using a protractor if available, but he did not relate a protractor to the partitioning of an arc length.

In response to the preinterview findings, the first two teaching sessions focused on developing a process for measuring an angle that entails quantifying the class of arcs subtended by an angle. Of relevance to the present study, Zac came to understand angle measure, regardless of unit, in terms of measuring *along* a subtended arc in a unit that conveys a fraction (or percentage) of the corresponding circle's circumference (e.g., an angle of one degree subtends 1/360 of any circle's circumference and an angle of one radian subtends $1/(2\pi)$ of any circle's circumference). Zac also conceived radian measure as the multiplicative relationship between the arc subtended by the angle and the corresponding circle's radius, where this multiplicative relationship holds for any circle centered at the vertex of the angle. By conceptualizing angle measure as the process of measuring and comparing various quantities (e.g., arc lengths, circumferences, and radii), Zac constructed meanings for degree and radian measures that had the same underlying structure, thus enabling him to transition flexibly between angle measure units. For instance, when given an angle measure of 35 degrees, Zac converted the measure to radians using the formula $35/360 = x/(2\pi)$. He explained that the two ratios must be equal because each angle measure corresponds to the same percentage of a circle's circumference, no matter the size of the circle.

Zac's actions over the course of the study suggested that he developed a preference for working with radian measures, as opposed to degree angle measures. For instance, Zac typically converted degree measures to radian measures at the outset of a problem, stating that he found radian measures easier to work with when solving a problem because of the explicit relationship to a circle's radius. Based on available research (Akkoc, 2008; Topçu et al., 2006), I did not expect Zac to adopt a preference for radian angle measures. I conjectured that Zac's preference for radian measure would support his construction of the sine function in ways that entailed measuring quantities in radii.

Constructing a Covariational Relationship

The fan problem (in Figure 2) and the fan applet (Figure 8) were designed to develop the sine function in the context of circular motion. The students spent the entirety of the third teaching session on the task. The problem statement did not include units for measuring the quantities, as I intended to determine if the students would spontaneously extend reasoning about the radius as a unit. I also chose the phrase *vertical distance above* to bring about the idea of a directed length with positive and negative measures representing positions above and below the horizontal diameter, respectively. The fan applet enabled the user to vary the position of the bug on the fan, vary the radius of the fan, show or hide the path traced out by the bug, show or hide a Cartesian coordinate system with the radius set as the unit of measure and the center of the fan as the origin, and show or hide the relevant measures in feet or radii. To begin the task, the applet only displayed points representing the starting position of the bug and the fan's center.

I first asked the students to describe the posed situation, and Zac responded that the bug's path traced out a circle. He added that the distance the bug traveled on this path is measurable in a number of radii, and he connected the bug's motion to an angle measurable in radians. I then animated the bug so that it traveled counter-clockwise while tracing out a highlighted path (Figure 8), and I asked the students to graph the relationship described in the problem statement. Zac conjectured that the graph is "like a sine or cosine" and mimicked the shape of a sinusoid with the

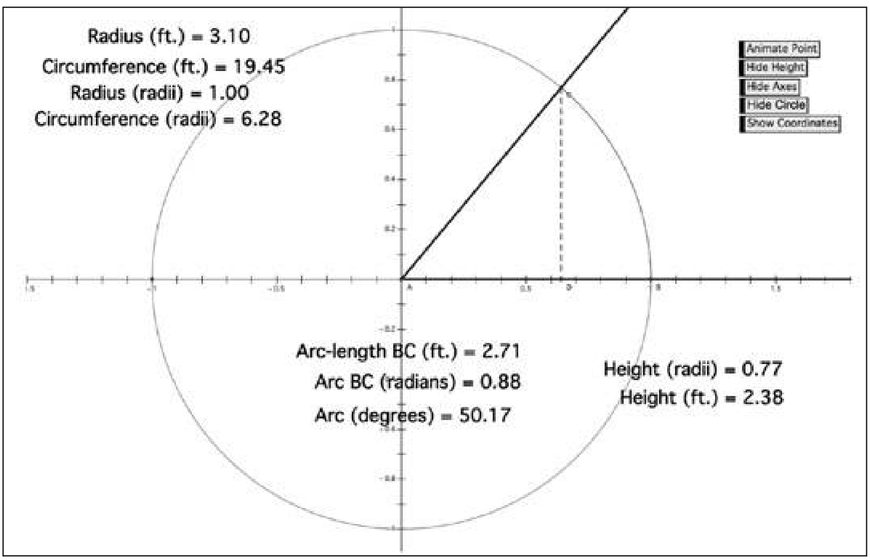

Figure 8. The fan applet with all information displayed.

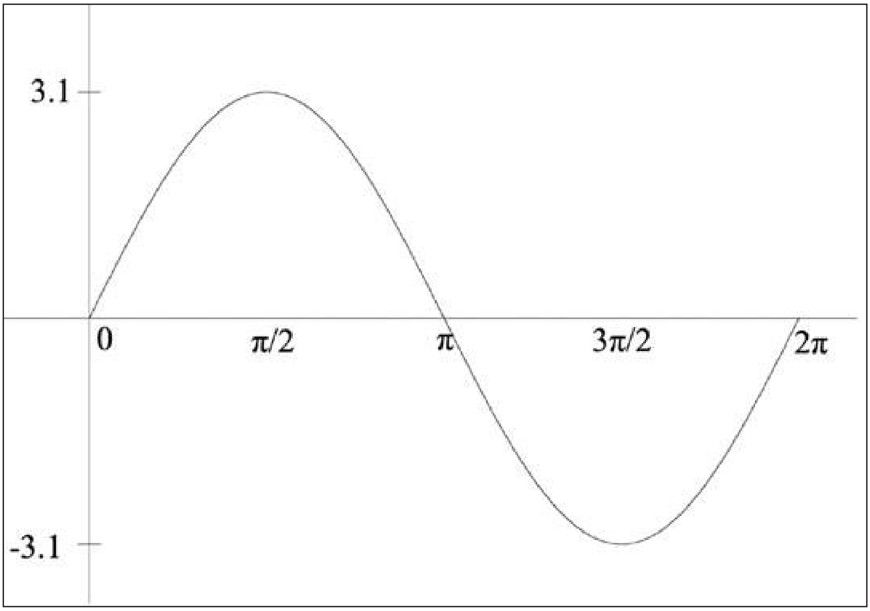

Figure 9. The students' graph for the fan problem restricted to one.

tip of his pen. Zac then led the other students in creating a visually correct graph (Figure 9) with the output measured in feet and the input measured in radians.[4] To justify their graph, Zac first described the directional covariation (MA2) of the two quantities (Table 1).

After Zac provided the description in Table 1, I asked the students to explain the "shape," or curvature, of their graph. The students again described the directional covariation (MA2) between the two quantities, and when prompted to explain further, they referenced the "continuous" rotation and circular shape of the fan. To the students, the graph was smooth because the ride is smooth both in motion and in shape.

Table 1
Directional Covariation

1	Zac:	At zero, it's going to be zero. Then a fourth of the way [*tracing a*
2		*quarter circle in the air with his pen*], it's going to be at the highest it
3		could be … then at pi, halfway, it's at zero again. Then three-fourths
4		of the way, three-halves pi, you have the lowest point it can be. Then
5		when you get to two pi, you're back at zero. Increases, decreases,
6		decreases, increases [*referring to each quarter of a revolution*].

[4] Zac used the terms *radii*, *radians*, and *radius lengths* interchangeably to refer to any measure that could be conceived as relative to the radius of a circle, whether the circle was specified or not.

Although the students' graph technically represents changing rates of change, the students' explanations did not reflect reasoning about rates of change (MA4/5) or amounts of change (MA3). In an attempt to generate a discussion about the concavity of their graph that focuses on the covariation of the two quantities, I created a graph (Figure 10) composed of three linear segments and asked the students to convince me that their graph is the correct graph. In response, Zac claimed that my graph conveys the same directional covariation (MA2) as his graph, but that my graph conveys that the "vertical distance is increasing at a constant rate" (MA5). He continued by explaining that a constant rate of change would imply, "Like for every one unit the total distance moves [*referring to the arc length swept out by the fan*] or increases, the vertical distance increases by a certain unit as well" (MA3).

Zac then suggested using the fan applet to compare changes in vertical distance for equal changes of distance traveled by the bug (MA3). We used the fan applet to determine and denote how the changes in the vertical distance change for equal changes of arc length over the first quarter of a revolution (Figure 11). After identifying that their graph conveyed the same covariational relationship as that constructed using the diagram (see Figure 3), Zac examined the covariational relationship over the second and third quarters of a revolution. Zac identified that the vertical distance approaches zero, but the magnitude of change of vertical distance increases (MA3) over the second quarter of a revolution. For the third quarter of a revolution, Zac described that the vertical distance "decreases at a decreasing rate" (MA5) and "the change in vertical distance is going to get smaller . . . concave up" for equal changes of arc length (MA3), apparently reasoning about

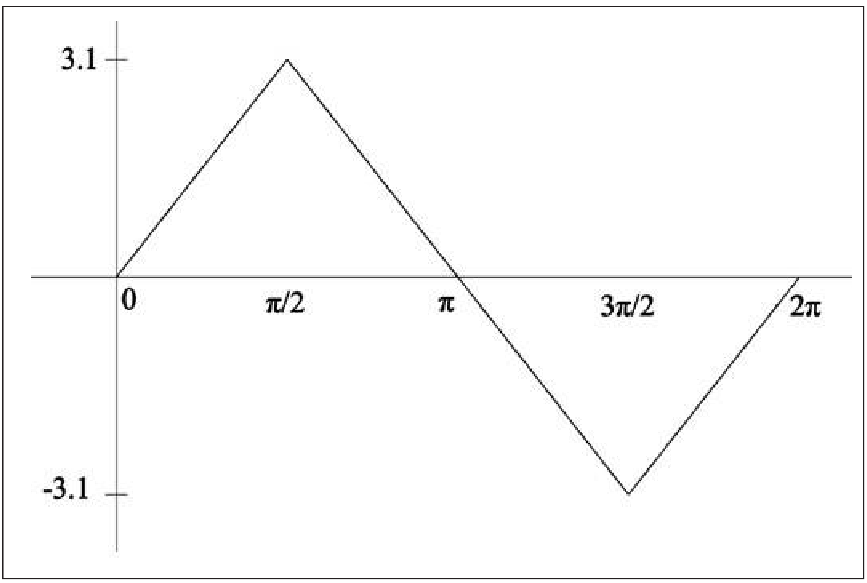

Figure 10. Researcher presented graph for the fan problem.

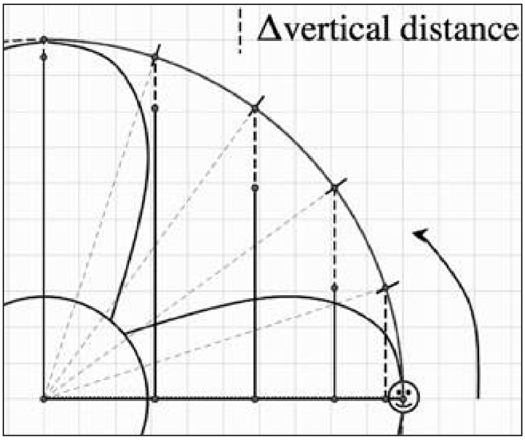

Figure 11. Representing amounts of change on the fan applet (reproduced diagram).

the magnitude of the change in vertical distance.

Zac's reasoning about amounts of change and rates of change might have drawn on knowledge from his calculus experiences, but it is worthwhile to note that he did not engage in these mental actions until I proposed a graph composed of linear segments. Instead, he described that the shape and continuous rotation of the fan explained the shape of the student-constructed graph prior to my posing of Figure 10. This is notable but unsurprising, as the research literature has suggested even high-performing students hold notions of rate of change that lack underpinnings in comparing amounts of change between two quantities (Carlson, 1998; Carlson et al., 2002). Despite Zac's previous experience in a calculus course, it would have been surprising had he spontaneously engaged in reasoning about amounts of change, even if he did show signs of rate of change reasoning.

Implications of Using the Radius as a Unit of Measure

After the students established that their graph conveyed the correct covariational relationship, I decided to press them on the units they chose for measuring each quantity. At this point in the lesson, no student had mentioned using the radius as a unit of measure for a quantity other than the distance traveled by the bug along its circular path. The students had measured the vertical distance in feet.

Because we had extensively discussed measuring arc lengths in both radii and more standard length units during the angle measure teaching sessions, I conjectured that the students were prepared to discuss the implications of measuring the vertical distance in a number of radii described in the above conceptual analysis. A particular focus of the angle measure teaching sessions was considering circles of various sizes and the multiplicative relationship that exists between the class of arcs subtended by an angle and each circle's radius (Moore, 2013). In an attempt to draw on these explorations, I suggested that we change the radius of the fan and reconsider their graph. After establishing that their graph must change if the radius of the circle is altered, I presented a graph that was identical to their graph except that the maximum and minimum output values were one and negative one, respectively. Without stating the units for the values on the vertical axis, I expressed to the students that although the graphs differ numerically, the graphs convey the same covariational relationship. I asked the students, "What units am I using now?" Zac immediately suggested, "A radian . . . output . . . it's radians," and drew attention to the bug being at most one radius above or below the center of the fan.

Zac subsequently identified how to transition between a vertical distance in radii and a vertical distance in feet. He explained, "multiply the percentage of the radius [*the output on the posed graph*] by the radius length." Zac also concluded that the posed graph works "for any circle" and that when the coordinate system on the applet conveys measures in "radius lengths," the values can be used for a circle with a radius of any length. Thus, it appears that Zac's ability to reason about measuring a length relative to the radius enabled him to conceive of my graphed relationship as applying to any circle.

Other than Zac's initial reference to "a sine or cosine," no individual had mentioned the sine function up to this point in the teaching session. My decision to not allude to the sine function was intentional. I had hoped that a graph of the sine function would emerge from the students' activity of coordinating covarying quantities and units of measure (with my support) rather than from their attempts to recall past knowledge of the sine function. Upon completion of the previous interactions, I defined the graph I produced as the sine function and introduced $f(\theta) = \sin(\theta)$ as the analytic notation that formalizes the covariational relationship between the vertical distance and angle measure in radii and radians, respectively. Zac concluded the teaching session by claiming that their graph corresponded to the formula $g(\theta) = 3.1\sin(\theta)$, reasoning that the output of the function f could be thought of as a fractional amount of the radius, and thus the output to g should be the same fractional amount of the radius measure in feet.

Evaluating the Sine Function

The explorations during the teaching session described above and the construction of the cosine function during the following teaching session did not emphasize evaluating these functions for specific input values other than 0, $\pi/2$, π, $3\pi/2$, and 2π radians. I avoided an emphasis on specified numerical pairs of values, as I intended that the students develop the capacity to reason about indeterminate values when constructing the covariational relationships defined by the sine and cosine functions. My intention was for the students to develop the capacity to hold in mind a meaning for the output of the sine function as a measure relative to the radius without having a specified value at hand. The sine and cosine functions cannot be evaluated using arithmetic operations, and thus it is important that students develop the ability to anticipate outputs of the sine and cosine functions (K. Weber, 2005).

In transitioning to evaluating trigonometric functions, I intended that the students come to understand the act of evaluating these functions as representing an instantiation, or *freezing*, of the covariation in order to determine a pair of specified values. During the fourth teaching session, I asked Zac to solve the positions on a circle problem to explore such a setting (Figure 12). The task also required considering the radius as a unit of measure.

> A certain arctic village maintains a circular cross-country ski trail for the enjoyment of its citizens during the winter months. Their trail has a radius of 2 kilometers. A certain skier started at position (2,0) one morning, skiing counterclockwise for 2.2 kilometers, where he paused for a brief rest. Determine the ordered pair that identifies the location where the skier rested.

Figure 12. The positions on a circle problem.

Zac first used a diagram to identify the starting position of the skier and the distance traveled by the skier (Figure 13). After determining a solution to the problem, Zac explained, "I found it easier to find the percentage of the radius first," and presented his solution (Table 2).

Zac's actions illustrate that he conceived the sine and cosine functions as relating input and output values measured relative to the radians.[5] This, in combination

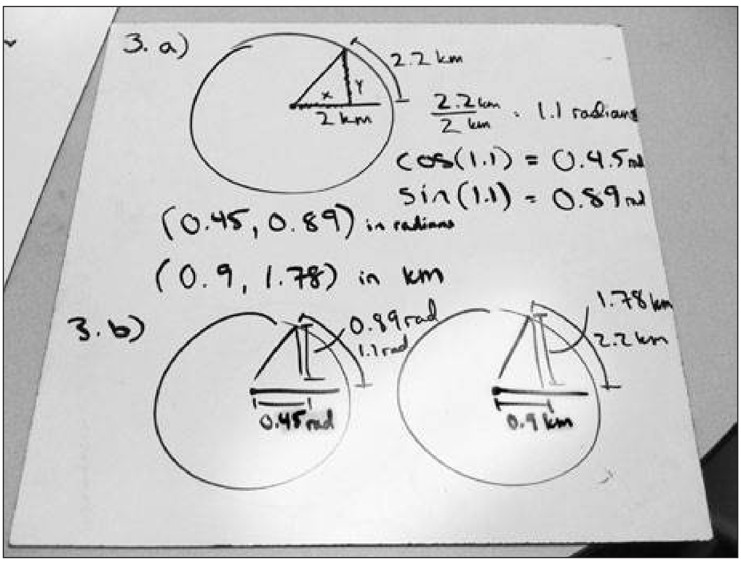

Figure 13. Zac's positions on a circle diagram.

[5] Zac often used the term *radian* to refer to any measurement relative to a circle's radius. I raised that the radian technically refers to an equivalence class of arcs (e.g., an angle measure) and that measures for a particular circle are given in radii, but Zac did not adhere (in words) to this distinction throughout the teaching experiment. His tendency to use the term *radian* might have stemmed from his understanding that "radian" measures apply to any circle. But, his misuse was only in terminology, as in activity he did coordinate when he was discussing a particular circle versus when he was discussing an arbitrary circle.

Table 2
Positions on a Circle Solution

Zac:	OK, so first I found out how many radians it took for him to get there [*tracing the traversed arc*], which was one point one, 'cause I took two point two kilometers divided by the radius, two kilometers, and got one point one radians [*pointing at the values on the board throughout his description*]. Then I took the cosine of one point one and got four point five radians [*meaning 0.45*], which gave me the horizontal distance [*tracing distance on his diagram*]. And I took the sine of one point one, and uh, got point eight nine radians, which gave me the vertical distance [*tracing distance on his diagram*], shown here as a percentage of a radius [*pointing to coordinate pair*], ordered pair in radians, percentage of a radius. So I got point four five, point eight nine. And then I multiplied those by the two kilometers to get it in kilometers [*pointing to coordinate pair given in kilometers*]. I got point nine, one point seven eight.
KM:	So why'd you multiply by the two kilometers?
Zac:	Uh, to get the radius. Because it's in a percentage of a radius, you have to multiply it by the radius so then I'll get kilometers.

with his ability to flexibly transition between units of measure, supported him in evaluating the sine and cosine functions in the context of circles other than a circle with a given radius of one. For instance, Zac determined the arc measure relative to the radius in order to obtain the input value necessary for the sine and cosine functions. Also, despite the lack of focus on evaluating trigonometric functions at specified values during the teaching session, Zac fluently used the trigonometric functions to relate specific input–output pairs.

Further Investigations of Circular Motion

The positions on a circle problem concluded the teaching sessions. During the following interview session, I used the Ferris wheel problem (Figure 14) to investigate Zac's covariational reasoning and ability to use the sine function (including using the radius as a unit) in a situation that presented a starting position and a vertical distance differing from those explored during the teaching sessions.

> Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full rotation. April boards the Ferris wheel at the bottom and begins a continuous ride on the Ferris wheel. Sketch a graph and determine a formula that relates the total distance traveled by April and her vertical distance from the bottom of the Ferris wheel (assume this is at ground level).

Figure 14. The Ferris wheel problem.

Zac oriented himself to the situation by drawing a diagram (Figure 15), tracing a portion of the circle to imagine April's trip, and determining the circumference of the circle (72π feet). He proceeded to draw a larger diagram of the Ferris wheel (Figure 16) and describe how the two quantities change together as the Ferris wheel rotates (Table 3).

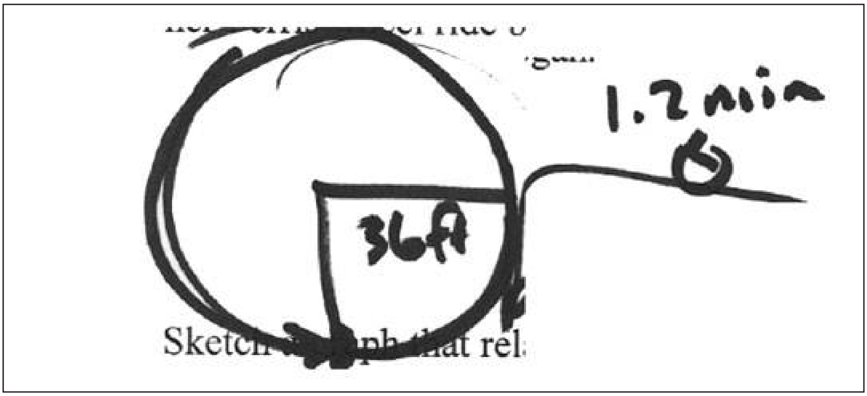

Figure 15. Zac's diagram of April's trip.

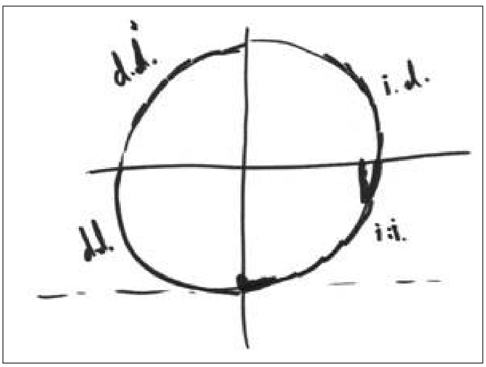

Figure 16. Zac's covariation of quantities on a diagram.

Although this problem included a novel starting position and involved tracking the rider's vertical distance from the bottom of the Ferris wheel (versus the rider's distance above the center of the Ferris wheel), Zac's actions illustrate how developing an image of how two quantities vary in tandem can support conceiving a graph as an emergent relationship between quantities. Contrary to his actions during the third teaching session, Zac discussed the relationship between the vertical distance (from the ground) and a traversed arc length using a diagram of the situation *prior* to creating a graph to represent this relationship. When using a diagram of the situation, Zac engaged in several ways of reasoning that he exhibited during the teaching session, including directional change (MA2), amounts of

Table 3
How the Two Quantities Change Together

1	Zac:	OK. So a really easy way to do this is divide it up into four quadrants [*divides*
2		*circle into four quadrants*]. 'Cause we're here [*pointing to starting position*],
3		for every unit the total distance goes [*tracing successive equal arc lengths*], the
4		vertical distance is increasing at an increasing rate [*writing i.i.*]. Then, uh,
5		once she hits thirty-six feet, halfway up, it's still increasing but at a decreasing rate
6		[*tracing successive equal arc lengths, writing i.d.*]. Uh, then when she hits the
7		top, at seventy-two, it's decreasing at an increasing rate [*tracing successive*
8		*equal arc lengths, writing d.i.*]. And then when she hits thirty-six feet again it's
9		still decreasing [*making one long trace along the arc length*], but at a
10		decreasing rate [*tracing successive equal arc lengths, writing d.d.*].
11	KM:	OK, so in terms of this quadrant [*pointing to the bottom right quadrant*], could
12		you show me how you know it's increasing at an increasing rate?
14	Zac:	So like, she moves that much there [*tracing an arc length beginning at April's*
15		*starting position*], that much here [*tracing an arc of equal length over the last*
16		*portion of April's path in that quadrant*], uh, the vertical distance there
17		changes by that much [*tracing vertical segment on the vertical diameter*],
18		which is really hard to see with this fat marker. And then, uh, the vertical
19		distance here changes by that much [*tracing vertical segment from the*
20		*starting position of the second arc length*], which is a much bigger change.
21		[*Zac proceeds by drawing a graph, Figure 17, while repeating his description*
22		*of how the two quantities varied in tandem.*]

change (MA3), and rate of change (MA5) thinking. For instance, to justify his rate of change description, he conceived of amounts of change in the vertical distance and considered variations in these changes for equal changes of distance traveled (MA3)—see Figure 16 and lines 14–20 in Table 3. By engaging in covariational reasoning, Zac regarded a novel situation as not so novel.

Zac then moved to creating a formula that corresponded to his graph. Zac first rotated his diagram of the situation counterclockwise by 90 degrees and explained, "Then I can actually make sine work." He paused for several seconds and rotated his diagram back to its original orientation (Table 4).

Zac determined the formula

$$vd = f(Td) = \sin\left(\frac{(Td - 56.5)}{36\text{ ft}}\right)$$

by the completion of this interaction, which (from an observer's perspective) defines a different relationship than the one shown in his graph. In creating this formula,

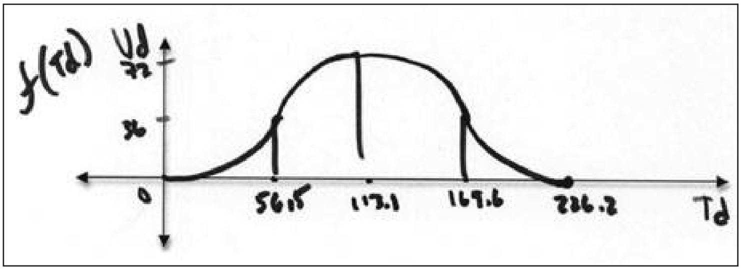

Figure 17. Zac's Ferris wheel graph.

Zac initially encountered a perturbation because of April's starting position, and I took his initial rotation of the diagram as an attempt to move April's starting position to the starting position used in the previous teaching sessions (3 o'clock). Ultimately, he reconciled the novel starting position by conceiving the position of the rider as measureable from the 3 o'clock position *along* the circumference of the Ferris wheel (lines 1–5). It is important to point out that Zac also noted that he needed this quantity's measure in a number of radii (lines 15–16), suggesting that he had conceptualized the argument of the sine function as a directed arc length measured in radii (or an angle measure in radians) from the 3 o'clock position.

At this time, Zac's formula was not numerically equivalent to his graph. Zac's formula can be thought of as representing the vertical distance from the horizontal diameter, although I was unsure if Zac was aware of this. Because Zac had not

Table 4
Determining the Vertical Distance

1	Zac:	[*diagram now in original position*] But I can still technically make it work
2		here just by taking, by making it, uh, the starting point [*pointing to the bottom*
3		*of the circle*] one sixty-nine point six feet around the circle. Or negative fifty-
4		six point five feet around the circle. So it's going backwards [*tracing arc*
5		*clockwise from the standard starting position to the bottom of the circle*].
6	KM:	OK
7	Zac:	I still get the vertical distance that way. Um, so ya [*pause*] So [*long*]
8		*pause*], that means, since I'm doing that, that means whatever the vert-, or
9		total distance is, I have to subtract fifty-six point five from it.
10	KM:	OK
11	Zac:	So let's see, vertical distance [*pause*] is equal to *f* of total distance [*writing*],
12		which is equal to total distance minus fifty-six point five [*writing*], which will
14		get me there [*pointing to the bottom of the circle*]. [*Pause*] And how [*inau-*
15		*dible*], divided by thirty-six feet to get me radians. [*Pause*] And then I take the
16		sine of that. So sine, so that will give me this [*referring to the graph*].

specified the output of the function in terms of a distinct vertical distance (e.g., from the ground or from the horizontal diameter) or unit of measure (e.g., feet or radii), I asked Zac to once more explain his formula. He responded that the sine function outputs "the vertical distance, [*pausing for a few seconds*] a percentage of the radius length, which I then need to multiply by thirty-six." Zac adjusted his formula to represent an output measured in feet,

$$vd = f(Td) = 36\sin\left(\frac{(Td - 56.5)}{36 \text{ ft}}\right),$$

a unit that was compatible with the vertical axis unit on his graph.

As the discussion continued, Zac noticed an inconsistency between his formula and his graph that stemmed from fluctuating between the vertical distance from the ground and the vertical distance form the horizontal diameter (Table 5). By considering a specific pair of values, Zac identified that his formula resulted in a value of –36 for an input of zero, which was not compatible with his graph. Zac reconciled this inconsistency by returning to his diagram and conceiving a measureable difference between the relevant vertical distances and then altering

Table 5
Correcting His Formula

Zac:	Uhhh, zero. Oh, OK, ya, I was thinking about that actually. So you're gonna have to change the formula a little bit, because you're gonna have to add seventy-two, or no, because this problem, from point zero, should give you a negative number. Negative thirty-six. I'm right there [*pointing to the bottom of the Ferris wheel*]. So then add thirty-six [*adding thirty-six to formula*].
KM:	OK, so why add thirty-six?
Zac:	[*working*] Because, um, technically right there it should give you zero [*pointing to the bottom of the Ferris wheel*]. Uh, but the problem's going to give you negative thirty-six because the value of sine theta [*tracing segment from center of the Ferris wheel to the bottom of the Ferris wheel*] at this point is negative one. Or sine, radians, one radian is negative one right there. And since you have to multiply it by the radius, it gives you negative thirty-six.
KM:	OK.
Zac:	So then to cancel that out, you just add thirty-six to it. Which would make sense too because then, uh, when you hit this point [*pointing to the 3 o'clock position*], technically that would be zero, so if you're adding thirty-six, that means thirty-six feet above the ground. And then at this point [*pointing to 12 o'clock position*], instead of just being thirty-six, because, you know, it's thirty-six from there [*tracing segment from center of the Ferris wheel to the top of the Ferris wheel*], it'd be seventy-two [*tracing segment from the bottom to the top of the Ferris wheel*], the whole distance from the ground.

his formula to convey this relationship. Compatible with his graph, which emerged from his image of the situation, his formula

$$vd = f(Td) = 36\sin\left(\frac{(Td - 56.5)}{36\text{ ft}}\right) + 36$$

emerged through an iterative process of considering the quantities of the situation, various units of measure, and how these quantities related to the particular relationship that Zac had conceived as the sine function.

Extending to Right Triangles

To conclude the interview session, I asked Zac to solve the Empire State Building problem (Figure 18). Considering that Zac's previous course work included some study of trigonometric functions, I was unsure if Zac would conceive a right triangle context in ways compatible with the ideas explored during the prior sessions or if his solution would stem from understandings constructed during previous course work.

> While site seeing in New York City, Bob stopped 1000 feet from the Empire State Building and looked up to see the top of the Building. Given that the angle of Bob's site from the ground was 56 degrees, determine the height of the Empire State Building.

Figure 18. The Empire State Building problem.

Zac first constructed a diagram of the situation and labeled the given values (Figure 19). He then explained, "From the circle, or triangle, we can determine that cosine of fifty-six degrees is equal to one thousand feet [writing corresponding equation] … one thousand feet is equal to the radians [referring to cos(56°)], because cosine fifty-six degrees is determined in radius lengths." During this explanation, Zac wrote the equation cos(56°) = 1,000 ft, which he later changed to cos(0.98) = 1,000 ft after converting the angle measure to radians (Figure 20). Zac's equation is numerically incorrect, but his description (and actions described below) suggests that he interpreted 1,000 and cos(56°) to represent equivalent measures of a quantity, with the measures representing a number of feet and radii, respectively. That is, Zac understood each measurement to correspond to an equivalent magnitude (that leg adjacent to the angle).

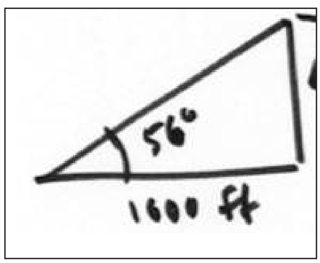

Figure 19. Zac's initial diagram for the Empire State Building problem.

After converting the given angle measure to a number of radians, Zac continued his solution(Table 6). Whereas Zac's initial equation

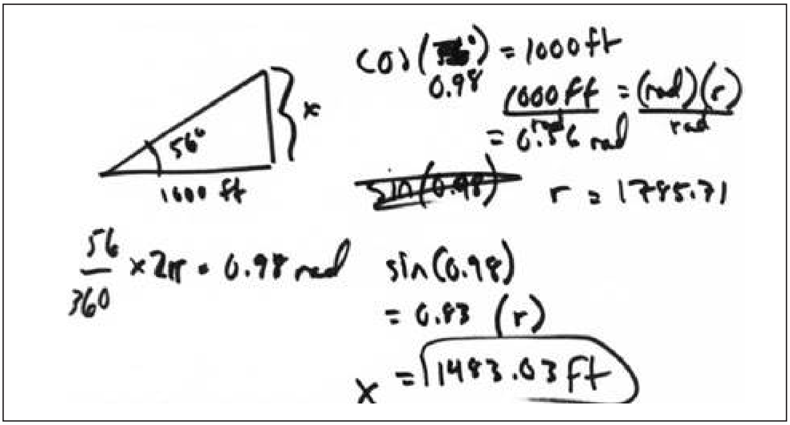

Figure 20. Zac's work on the Empire State Building problem.

of cos(56°) = 1,000 ft represented equivalent measures in different units, Zac used the equation 1,000 ft = (*rad*)(*r*) to represent that the person's distance from the building in feet is *rad*, or cos(0.98), times as large as the denoted radius (the person's distance from the top of the building). Zac's claim that he needed to "figure out what cosine point nine eight is actually equal to" also indicates that he was aware that his initial equation was not numerically correct. As Zac continued, an important feature of his solution was that he anticipated evaluating trigonometric functions. For instance, he anticipated evaluating cos(0.98) to determine the person's distance from the building in radii, a value that enabled him to determine the radius length in feet. Then, without determining a numerical value for the radius, Zac conceived sin(0.98) as representing the height of the building measured as a "percentage of a radius length" before evaluating the expression. Following Zac's explanation, he executed each anticipated calculation to obtain a correct answer of approximately 1,483.03 feet (Figure 20).

Table 6
Solving the Empire State Building Problem

Zac:	A thousand feet is equal to the radians times the radius length [*writing* "(*rad*)(*r*)"], or *r*.
KM:	OK.
Zac:	OK, 'cause the radians is just a percentage of the radius length. OOOK. So, now what I want to do is figure out what cosine point nine eight is actually equal to [*pointing to rad*], and using that I can find out what the radius length is [*pointing to r*]. So then when I do sine of point nine eight, I already know what the radius length is, so when I get the answer to that [*referring to* sin(0.98)] all I have to do is multiply by the radius length and I'll get that part [*identifying the vertical segment on his triangle*].

In all, Zac's solution hinged on reasoning about measuring quantities as a fraction of the radius to determine the "radius length" and the height of the building. Specifically, Zac conceived quantities as simultaneously measurable in radii and other length units, with radii measures conveying that measures in other length units are so many times as large as the length of the radius in that unit. Zac's understanding of the radius as a unit of measurement and the proportionality that holds across all length-radius measure pairs (such that the measures are in the same unit) enabled him to anticipate obtaining and performing actions on output values of the sine and cosine functions prior to evaluating these functions and executing calculations.

Up to this point in his solution process, Zac had not drawn a circle or radius on his diagram. I asked Zac to further describe his meaning of "the radius," and he responded by constructing a circle and then a right triangle within the circle (Figure 21). After labeling the right triangle with each determined and unknown value, Zac described the hypotenuse of the right triangle as "the radius." Zac then explained his decision to construct a circle (Table 7).

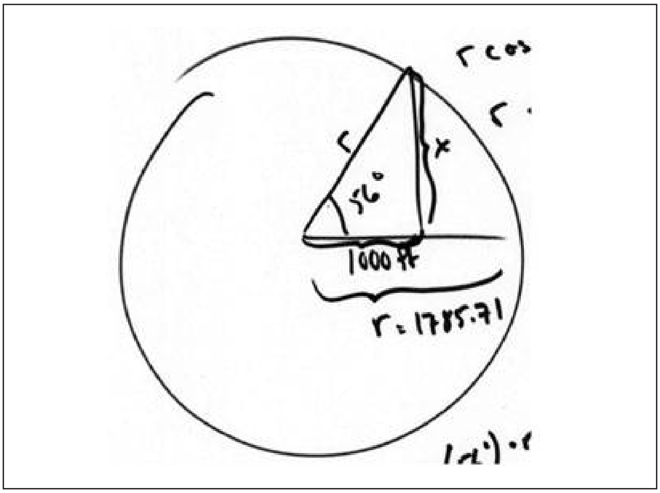

Figure 21. Zac's refined diagram on the Empire State Building problem.

Table 7
Benefits of Using Circles

Zac:	To make it easier to understand. How I was originally taught was just with triangles. Now that we've started using circles it makes a whole lot more sense.
KM:	So could you say a little bit about why it makes a little more sense now?
Zac:	Uh, because I always just thought hypotenuse was, you know, just that side of a triangle. You could use Pythagorean's Theorem to find out what it was very easily. And now that we've figured out, now I'm looking at it and seeing it's the radius, it makes a lot more sense to be able to find the horizontal and vertical distances according to the radius [*waving tip of pen across the radius*].

Zac's explanations suggest that the study's approach to trigonometry was novel to him and that he found value in "using circles" with the sine and cosine functions (lines 1–2). He now understood the hypotenuse as related to a circle's radius and as a length that can be used to measure other lengths, as opposed to "just that side of a triangle." Not only did Zac's solutions during the teaching and interview sessions suggest that reasoning about the radius as a unit of measure provided him with a foundational way of thinking, but these statements convey that he was aware of and valued thinking about measuring lengths according to another length. To Zac, measurement formed a foundational and coherent way to conceive the use of trigonometric functions in both circle and triangle contexts.

Discussion

Zac's progress over the course of the study was rather fluid and, as such, illustrates that he constructed a system of trigonometric meanings involving angle measure, quantity, measurement, and covariation that incorporated both circle and triangle contexts. In this section, I discuss aspects of Zac's meanings for the sine function against the backdrop of reasoning about quantities including angle measures and directed lengths. I also highlight Zac's capacity to reason about magnitudes and measuring in radii and examine general themes in his meanings for the sine function.

Covariational Reasoning, Angle Measures, and Arcs

Returning to the first guiding research question for the study, Zac developed a meaning for the sine function that entailed coordinating various mental actions associated with covariational reasoning. When beginning the teaching experiment, Zac's actions did not indicate reasoning about rates of change (MA5) or amounts of change (MA3). Although the curvature of his original graph (Figure 9) technically represented changing rates of change, Zac described the curvature of his graph in terms of the shape and continuous motion of the fan. Such activity is compatible with reports (Carlson, 1998; Monk, 1992; Monk & Nemirovsky, 1994) of students relying on nonquantitative features when relating graphs to corresponding physical situations. A consequence of this orientation toward graphing is that students do not conceive graphs as emergent representations of covarying quantities and instead approach graphs as objects in and of themselves (P. W. Thompson, 1994c; E. Weber, 2012).

Zac's transition from relying on nonquantitative features of the physical situation to reasoning about how quantities vary in tandem stemmed largely from my prompting during the teaching sessions. Zac first engaged in several of the covariation mental actions when contrasting the relationships presented by multiple graphs (Figures 9–10), which suggests that my posing of Figure 10 formed a *generative alternative* (Rasmussen & Marrongelle, 2006). However, during the interview in my last session with Zac, he coordinated direction of change, rates of change, and amounts of change (including comparisons in amounts of change) between two quantities to create a graph independent of my guidance. This suggests that these mental actions had become aspects of his sine function meanings.[6]

Zac's ability to work with a starting position and vertical distance that differed from those explored during the teaching sessions illustrates a potential benefit of developing meanings that include a structure of covarying quantities: Namely, despite the novel starting position and vertical distance represented in the interview task, Zac understood the situation as entailing a covariational relationship like that of the sine function and was thus able to assimilate the situation to his meaning for the sine function. I also note that as opposed to first creating a graph and then using his graph to describe the covariational relationship as he had during the teaching session, Zac initially reasoned about rates of change and amounts of change between the relevant quantities using a diagram of the situation. His graph then emerged in a way that captured his conceived relationship. Such actions echo Moore and Carlson's (2012) emphasis on the interplay between a student's image of a situation and his or her ability to generate mathematical formalisms (e.g., graphs and formulas) that are representative of some invariant aspect of the situation (e.g., a quantitative relationship). With meanings tied to covariational reasoning and an approach that first focuses on the situation at hand, any starting position or vertical distance can be examined using mental actions of the same nature.

Determining how to engender and leverage covariational reasoning remains a critical area of need in mathematics education (Carlson et al., 2002; Oehrtman et al., 2008), and thus a natural question is: What might have supported Zac's covariation meaning for the sine function? Returning to the second guiding research question of the study, one possible support is that Zac's meanings for angle measure foregrounded a dynamic process involving the act of measuring *along* an accumulated arc. By imagining measuring the distance an object travels *along* an arc in a number of radii, Zac conceived circular motion in terms of a varying quantity—angle or arc measure—and subsequently coordinated a second quantity with a varying arc. It is questionable whether or not Zac's prior meanings for angle measure, which involved understanding measures as little more than labels or pointers to geometric objects, would have supported a covariation meaning for the sine function. As previously illustrated (Moore, 2012), a student who conceives angle measures in nonquantitative ways is likely to encounter difficulty constructing the sine function as a structure of covarying quantities.

An arc meaning for angle measure might also have influenced Zac's ability to assimilate a right triangle context to his covariation meanings for the sine function, hence producing an accommodation to his triangle trigonometry meanings. With angle measure meanings rooted in measuring arcs, angle measures can be thought of in terms of a circle even in the case that the angle measure is presented in a

[6] As Johnson (2012a) noted, comparing amounts of change (e.g., considering variation in the intensity of change) is a nontrivial act for students, and Zac's capacity to enact such reasoning unprompted during the interview cannot be overstated.

circle-less[7] right triangle. When Zac encountered a right triangle without an explicit circle, he immediately envisioned the hypotenuse of the right triangle as forming the radius of a swept out circle. It followed that the meanings Zac had constructed during the previous teaching sessions became applicable to a right triangle context, which enabled him to construct compatible meanings for the two contexts. Just as he had evaluated the sine and cosine functions in a circle context, Zac conceived a right triangle situation as an instantiation of the covariational relationships defined by these functions.

The Radius as a Unit of Measure and Magnitude Reasoning

Whereas students and teachers often have difficulty relating trigonometric functions to circle contexts, Zac did not encounter such difficulties, partially because of his capacity to reason about measuring lengths in radii. Conceiving a circle's radius as a viable unit of measure enabled Zac to view a circle of any given radius as representative of the unit circle; Zac understood that, when measured in radii, corresponding measures on all circles are numerically equivalent. By conceiving the input and output values of trigonometric functions as measures relative to a circle's radius, his meanings for the sine function incorporated any given circle. When circles were presented with quantities measured in a unit other than radii, Zac applied trigonometric functions by keeping in mind that their input and output values convey a multiplicative relationship between the measured magnitude (e.g., arcs and vertical distances) and the radius, and he used this relationship to transition between a circle's radius as the unit of measure and more standard length units (e.g., feet or meters) as needed.

Just as Zac conceived a circle's radius as a viable unit of measure, he conceptualized the hypotenuse of a right triangle as a unit for measuring the legs of that right triangle. This outcome stemmed from Zac conceiving a right triangle's hypotenuse as the radius of a circle, which supported him in assimilating right triangle contexts to the meanings developed during the teaching sessions. Zac came to understand that circle and triangle contexts involve using the sine and cosine functions to relate angle measures (or arc measures in radii) to a multiplicative comparison between two other lengths (e.g., the measure of a length relative to another length). K. Weber (2005) argued that students are better positioned to use trigonometric functions if they come to view the geometric objects of trigonometry as tools for reasoning about these functions. Zac's activity indicates that ideas of measurement, including reasoning about the radius as a unit of measure, support the construction of meanings that incorporate the unit circle and right triangle in productive and compatible ways. Returning to Brown's (2005) claim that an underlying difficulty in trigonometry is that students are required to reason about ratios in triangle contexts and real numbers in circle contexts, Zac's activity illustrates that this assumed divide is unnecessary from a measurement

[7] I note that a right triangle is not technically circle-less if an angle measure is labeled with an arc centered at the vertex of the angle, as the arc conveys at least a portion of a circle's circumference.

perspective. With a basis in measurement, the ratios and numbers that Brown referred to each represent measures. The trigonometric ratios typically found in right triangle trigonometry curricula represent using the right triangle's hypotenuse (which can be conceived as a circle's radius) as a unit of measure. Relative to a circle context, values associated with the unit circle represent measures in radii that hold for all circles; in the case that a particular circle under consideration is given with measures in a unit magnitude other than the radius, ratios emerge as representing radii measures.

Collectively, Zac's capacity to reason about magnitudes, work with undetermined values, and anticipate the execution of calculations materializes as an influence on his meanings for the sine function. Each of these aspects underscores an important facet of quantitative reasoning. At its foundation, quantitative reasoning involves activity that is not reliant on specified numbers or executing calculations (Moore, 2013; P. W. Thompson, 1990). One can conceive of and give meaning to the difference of two lengths without having the specific measures of these lengths available to determine this difference. Or, one can reason about a length measurement in radii without having to physically perform the measurement or obtain the measure through executing a calculation.[8] Likewise, one can envision performing operations on a radii measure prior to actually obtaining the measure. In Zac's case, he conceived a measure in radii (e.g., the output to the sine and cosine functions) as conveying a multiplicative relationship between a length and the circle's radius. The numerical value of this measure was secondary to his reasoning in that he did not need a specified numerical value to give meaning to the measure or anticipate subsequent actions involving the measure. If the solution so required, Zac obtained numerical values while holding in mind that these values represented quantitative relationships, as opposed to numbers devoid of a quantitative meaning. Zac's capacity to remain focused on values in terms of their quantitative meaning provides an explanation as to why Zac rarely lost track of the meaning of numbers and often worked fluently in quantitatively rich situations despite using technically incorrect notation (see his solution to the Empire State Building problem).[9]

Returning to the explorations during the teaching sessions, the approach to introducing the sine function might have supported Zac's development of meanings that entail reasoning about magnitudes and undetermined values. During the fan problem, I prompted Zac to use the diagram of the situation to identify and compare magnitudes (lengths of undetermined measures) for successive intervals of arc length. He was not given numerical measures for these magnitudes because I had intentionally kept all output values hidden on the fan applet. By having Zac consider magnitudes without available numerical values, he likely constructed

[8] Zac had a calculator at his disposal during the entire study. His disposition to anticipate calculations and their meaning, including how he might subsequently use the result of anticipated calculations, before using a calculator speaks to his engagement in quantitative reasoning.

[9] It is important to note that although Zac's use of notation was incorrect to an observer, Zac used the notation in sensible ways to represent relationships between quantities and their values (e.g., using an equation to represent equivalent measures but not numbers).

meanings for the sine function that were based in magnitude reasoning. That is, the covariational relationship that became a central component of his sine function meaning was not dependent on numerical values with a specified unit; the same covariational relationship existed regardless of the unit chosen to measure the quantities (e.g., the given unit or radii). P. W. Thompson (2011) argued that magnitude reasoning is critical for an understanding of quantity. Zac's actions provide evidence that magnitude reasoning, including understanding that quantities simultaneously have measures in different units, can play an important role in students' trigonometric meanings.

K. Weber (2005) noted that trigonometric functions are one of a student's earlier function experiences with output values that cannot be computed using arithmetic operations. Because of this, it is important that students develop meanings for trigonometric functions that are not reliant on having specified numerical values.[10] This is precisely the nature of Zac's magnitude-based meanings for the sine function, which speaks to the type of reasoning (e.g., a process concept of function) that several researchers have articulated as critical for developing a conceptual understanding of function (Dubinsky & Harel, 1992; Oehrtman et al., 2008). If necessary, trigonometric functions can be evaluated to determine specified measures, but it is important that students understand that a pair of specified measures represents an instantiation of the covarying quantities and that evaluating the function is not necessary to reason about the meaning of the pair of values.

Concluding Remarks

Although angle measure, covariational reasoning, and quantitative reasoning (including measurement) all played an important role in the present study, these ideas and ways of reasoning are not restricted to the study of trigonometry. Angle measure is often introduced during elementary school, and school mathematics is filled with *potential* opportunities to engender learning through quantitative and covariational reasoning. I emphasize the word *potential* because, as several researchers have noted, quantitative and covariational reasoning are currently not a fundamental part of school mathematics in the United States (e.g., Oehrtman et al., 2008; Smith & Thompson, 2007). Similarly, researchers have shown that common approaches to angle measure do not create foundational meanings for the study of trigonometry (e.g., Akkoc, 2008; Moore, 2012, 2013).

I took Zac's quantification of angle measure as a critical springboard for a covariational approach to the sine function. Whether or not Zac would have achieved similar progress without developing novel meanings for angle measure during the initial teaching sessions remains an open question. Future research should consider exploring a covariational approach to the sine function with a particular focus on characterizing students' angle measure meanings and the influence of these meanings

[10] Pushing for reasoning about unspecified values also explains why I avoided exploring the so-called *special angle* (π/3, π/4, π/6, etc.) and exact value pairs during the introduction to the sine function.

on their construction of the sine function. As reflected by the above approach to angle measure and the sine function, I also took measurement and variation as complex concepts central to the study. Zac showed rather sophisticated ways of reasoning in these respects, and the research base on trigonometry would benefit from additional studies that investigate a similar approach to trigonometry with students of varying levels of sophistication in their measurement and variation schemes. Likewise, the research base on trigonometry would benefit from studies that extend the current work to other populations, including preservice teachers, inservice teachers, and secondary students. As Zac likely had several experiences with trigonometric functions prior to the study, working with students being introduced to trigonometry for the first time forms an especially important area for future research. This article presents only one case (and one researcher's perspective) of a student constructing a coherent system of meanings for the sine function, and other students can and will develop meanings that differ from Zac's meanings. The findings of this study provide a detailed account of how quantitative reasoning can play an important role in the teaching and learning of trigonometry. How such an approach may be extended to other populations and settings remains an open question.

References

Akkoc, H. (2008). Pre-service mathematics teachers' concept image of radian. *International Journal of Mathematical Education in Science and Technology, 39*(7), 857–878. doi:10.1080/00207390802054458

Axler, S. (2009). *Precalculus: A prelude to calculus* (1st ed.). Hoboken, NJ: John Wiley & Sons, Inc.

Blackett, N., & Tall, D. O. (1991). Gender and the versatile learning of trigonometry using computer software. In F. Furinghetti (Ed.), *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 144–151). Assisi, Italy.

Bressoud, D. M. (2010). Historical reflections on teaching trigonometry. *Mathematics Teacher, 104*(2), 106–112.

Brown, S. A. (2005). *The trigonometric connection: Students' understanding of sine and cosine* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (Order No. 3233908)

Cai, J., & Wang, T. (2010). Conceptions of effective mathematics teaching within a cultural context: Perspectives of teachers from China and the United States. *Journal of Mathematics Teacher Education, 13*(3), 265–287. doi:10.1007/s10857-009-9132-1

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Issues in mathematics education: Research in collegiate mathematics education III* (Vol. 7, pp. 114–162). Providence, RI: American Mathematical Society.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45–75. doi:10.1007/s10649-005-0808-x

Castillo-Garsow, C. C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (Order No. 3425752)

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 55–73). Laramie, WY: University of Wyoming.

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (in press). Chunky and smooth images of change. *For the Learning of Mathematics.*

Clark, P. G., Moore, K. C., & Carlson, M. P. (2008). Documenting the emergence of "speaking with meaning" as a sociomathematical norm in professional learning community discourse. *Journal of Mathematical Behavior, 27*(4), 297–310. doi:10.1016/j.jmathb.2009.01.001

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66–86.

Doerr, H. M. (1996). Integrating the study of trigonometry, vectors, and force through modeling. *School Science and Mathematics, 96*(8), 407–418. doi:10.1111/j.1949-8594.1996.tb15864.x

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85–106). Washington, DC: Mathematical Association of America.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439–478. doi:10.1080/07370000701632397

Fi, C. (2003). *Preservice secondary school mathematics teachers' knowledge of trigonometry: Subject matter content knowledge, pedagogical content knowledge and envisioned pedagogy* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (Order No. 3097526)

Fi, C. (2006). Preservice secondary school mathematics teachers' knowledge of trigonometry: Cofunctions. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 833–834). Mérida, México.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A proceptual view of elementary arithmetic. *Journal for Research in Mathematics Education, 26*(2), 114–141.

Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 383–432. doi:10.1080/07370008.2010.511565

Hertel, J., & Cullen, C. (2011). Teaching trigonometry: A directed length approach. In L. R. Wiest & T. Lamberg (Eds.), *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1400–1407). Reno, NV.

Izsák, A. (2003). "We want a statement that is always true": Criteria for good algebraic representations and the development of modeling knowledge. *Journal for Research in Mathematics Education, 34*(3), 191–227.

Johnson, H. L. (2012a). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *Journal of Mathematical Behavior, 31*(3), 313–330. doi:10.1016/j.jmathb.2012.01.001

Johnson, H. L. (2012b). Two forms of reasoning about amounts of change in covarying quantities. In L. R. Van Zoest, J.-J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 143–150). Kalamazoo, MI.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington DC: Authors. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Meira, L. (1995). The microevolution of mathematical representations in children's activity. *Cognition and Instruction, 13*(2), 269–313. doi:10.1207/s1532690xci1302_5

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 175–193). Washington, DC: Mathematical Association of America.

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Issues in mathematics education: Research in collegiate mathematics education I* (Vol. 4, pp. 139–168). Providence, RI: American Mathematical Society.

Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 75–92). Laramie, WY: University of Wyoming.

Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics, 83*(2), 225–245. doi:10.1007/s10649-012-9450-6

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *Journal of Mathematical Behavior, 31*(1), 48–59. doi:10.1016/j.jmathb.2011.09.001

Moore, K. C., LaForest, K. R., & Kim, H. J. (2012). The unit circle and unit conversions. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 16–31). Portland, OR. Retrieved from http://sigmaa.maa.org/rume/crume2012/RUME_Home/RUME_Conference_Papers_files/RUME_XV_Conference_Papers.pdf

Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *Journal of Mathematical Behavior, 32*(3), 461–473. doi:10.1016/j.jmathb.2013.05.002

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27–42). Washington, DC: Mathematical Association of America.

Piaget, J. (2001). *Studies in reflecting abstraction* (R. L. Campbell, Ed. and Trans.). Hove, United Kingdom: Psychology Press Ltd. (Original work published 1977)

Rasmussen, C. L. (2001). New directions in differential equations. A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20*(1), 55–87. doi:10.1016/S0732-3123(01)00062-1

Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. J*ournal for Research in Mathematics Education, 37*(5), 388–420.

Skemp, R. (1979). *Intelligence, learning, and action*. New York, NY: John Wiley & Sons.

Smith, J., III, & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). New York, NY: Lawrence Erlbaum Associates.

Steckroth, J. J. (2007). *Technology-enhanced mathematics instruction: Effects of visualization on student understanding of trigonometry* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (Order No. 3282501)

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York, NY: Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillside, NJ: Erlbaum.

Stewart, J., Redlin, L., & Watson, S. (2012). *Precalculus: Mathematics for calculus* (6th ed.). Stamford, CT: Brooks/Cole, Cengage Learning.

Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Thompson, K. A. (2007). *Students' understanding of trigonometry enhanced through the use of a real world problem: Improving the instructional sequence* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (Order No. 3280913)

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebraic.* Unpublished manuscipt, Department of Mathematical Sciences and Center for Research in Mathematics and Science Education, San Diego State University, San Diego, CA.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: State University of New York Press.

Thompson, P. W. (1994b). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2–3), 229–274. doi:10.1007/BF01273661

Thompson, P. W. (1994c). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Issues in mathematics education: Research in collegiate mathematics education I* (Vol. 4, pp. 21–44). Providence, RI: American Mathematical Society.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–49). Morélia, Mexico.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOMe* (pp. 33–57). Laramie, WY: University of Wyoming.

Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education* (pp. 57–93). New York, NY: Springer.

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*, 415–432. doi:10.1007/s10857-007-9054-8

Topçu, T., Kertil, M., Akkoç, H., Yilmaz, K., & Önder, O. (2006). Pre-service and in-service mathematics teachers' concept images of radian. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 281–288). Prague, Czech Republic: PME.

Van Brummelen, G. (2009). *The mathematics of the heavens and the earth: The early history of trigonometry*. Princeton, NJ: Princeton University Press.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, DC: Falmer Press.

Weber, E. (2012). *Students' ways of thinking about two-variable functions and rate of change in space* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (Order No. 3503170)

Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal, 17*(3), 91–112. doi:10.1007/BF03217423

Author

Kevin C. Moore, Department of Mathematics and Science Education, University of Georgia, 105 Aderhold Hall, Athens, GA 30602; kvcmoore@uga.edu

Submitted March 7, 2013

Accepted August 7, 2013