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Using abstraction to analyze instructional tasks and their implementation

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A R T I C L E I N F O

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A B S T R A C T

Over the past few decades, Piaget's forms of abstraction have proved productive for developing explanatory models of student and teacher knowledge, yet the broader applicability of his abstraction forms to mathematics education remains an open question. In this paper, we adopt the Piagetian forms of abstraction to accomplish two interrelated goals. Firstly, we analyze instructional tasks to develop hypothetical accounts of the abstractions that might occur during students' engagement with them. Secondly, we draw on middle- and secondary-grades classroom data to discuss the abstractions that occurred during the implementation of those instructional tasks. Because this paper represents an initial attempt at extending the applicability of Piagetian forms of abstraction, we close with potential implications of such use and possible avenues for future research. Most notably, we highlight the complexities involved in supporting abstraction through curriculum and instruction.

1. Introduction

Piaget's (1970, 2001) genetic epistemology has played a critical role in mathematics education research. Researchers have adopted his theory to develop models of students' mathematics, teachers' mathematics, and student-teacher interactions. In doing so, researchers have identified meanings that prove productive for students' mathematical development, as well as meanings that constrain students' mathematical development (e.g., Carlson et al., 2002; Ellis et al., 2020; Ellis et al., 2015; Hackenberg, 2010; Norton & Wilkins, 2009; Paoletti, 2020; Steffe & Olive, 2010; Thompson, 1994a; Tillema, 2014). These researchers have also provided useful ways to characterize teaching in terms of how teachers' meanings influence their instruction and student interactions, including the ways in which teachers construct, learn from, and build upon students' ways of thinking (e.g., Liang, 2021; Silverman & Thompson, 2008; Simon, 1995; Steffe & D'Ambrosio, 1995; Tallman, 2015; Tallman & Frank, 2020; Teuscher et al., 2016).

Abstraction has emerged from Piaget's genetic epistemology as a useful construct to formulate differentiated accounts of knowledge development in a diverse number of contexts (e.g., Battista, 2007; Ellis, Lockwood, & Ozaltun-Celik, 2022; Ellis, Paoletti et al., 2024; Moore, 2014; Simon et al., 2010; Tallman & O'Bryan, 2024; Thompson, 1994a). Given the usefulness of Piaget's perspective on abstraction for developing accounts of student and teacher cognition, we hypothesized that these constructs may prove productive for

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analyzing other factors that contribute to students' mathematical experiences. A key aspect of mathematics education work is, after all, pushing and extending theory into new areas in order to see what insights might be formed.

Curricular materials represent one of the more critical artifacts, resources, or objects that affect teaching and, hence, student learning (Howson et al., 1981; Kilpatrick, 2011; Remillard, 2005; Thompson, 2013). We thus extend Piaget's forms of abstraction in order to provide accounts rooted in a mental action framing for curricular materials and their implementation. We investigate the research question, "In what ways does abstraction provide a viable tool to characterize potential learning as it relates to instructional tasks and their implementation?" We first provide a brief discussion of mathematical tasks and introduce Piaget's forms of abstraction as a framework for analyzing task design and implementation. After this synthesis, we describe the project setting in which we collected instructional tasks and their implementations. We also describe our analysis process for using Piaget's forms of abstraction to characterize the materials and their implementation. We discuss analysis results by focusing on a subset of the collected instructional tasks and presenting their implementation. Our results illustrate how the forms of abstraction provide a grounding by which to differentiate potential learning outcomes, as well as the potential intricacies involved in supporting students' abstraction processes.

2. Background

Doyle (1988) introduced the notion of an academic task to provide a "treatment theory to account for how students learn from teaching" (p. 167). Doyle's (1983) framing of academic tasks revolved around products to be produced by students, the operations students might use to produce those products, and the resources available to students to support that production. Since Doyle laid the foundation for viewing curriculum as a collection of academic tasks, mathematics educators have pursued linkages between teaching and learning as it relates through task design and implementation (e.g., Remillard et al., 2009).

2.1. Task analysis and implementation

Stein et al. (1996) introduced the notion of a *mathematical task* as an academic task that is situated around a particular mathematical idea or concept. Stein et al. (1996) summarized the importance of tasks and their implementation by describing them as "the proximal causes of student learning from teaching" (p. 459), and they presented Fig. 1 as a way to frame the different phases of a mathematical task. Fig. 1 captures that a mathematical task is best viewed as a dynamic object, evolving from design to implementation with a collection of influencing factors that have since formed focal points for researchers. No single account of a mathematical task can capture all factors contributing to student learning, yet mathematical tasks (and more broadly curricular resources) are worthy of study due to their significant influence on students' learning opportunities and mathematical experiences (Kilpatrick, 2011; Remillard, 2005; Remillard et al., 2009; Stein et al., 2007; Stylianides & Stylianides, 2008).

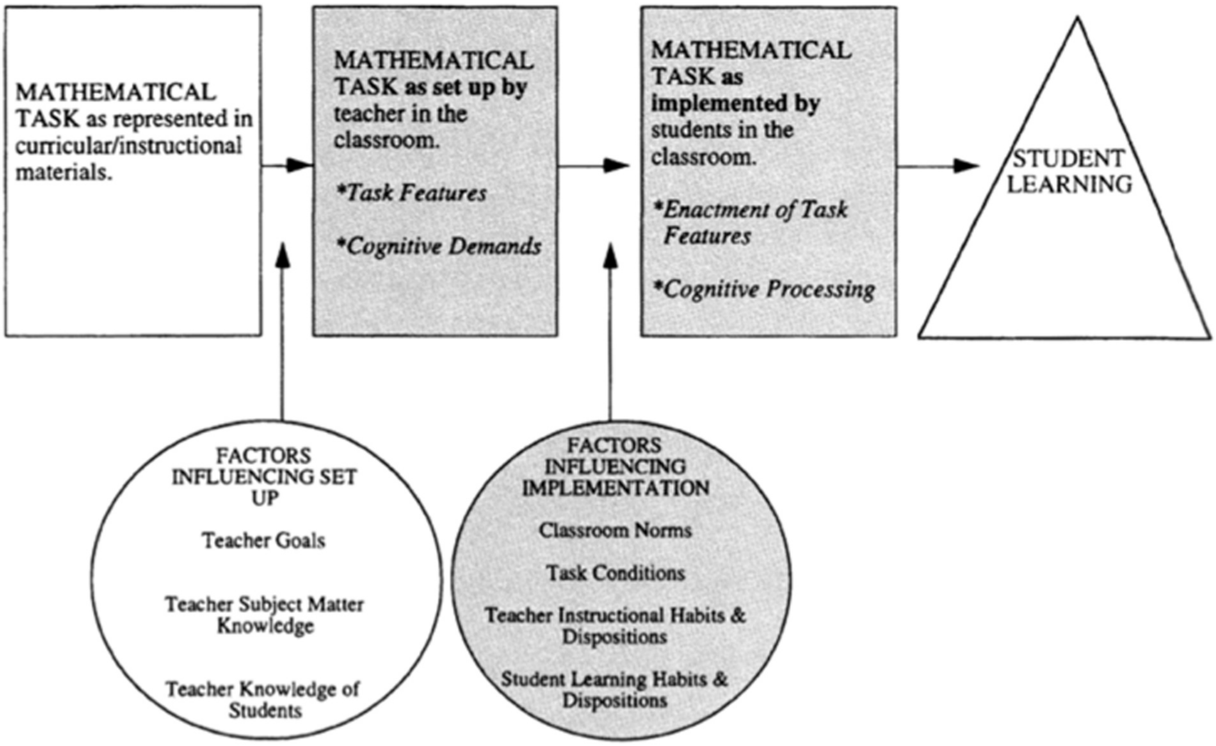


Fig. 1. Different phases and aspects contributing to mathematical tasks. Reproduced from Stein et al., (1996, p. 459).

The phases of mathematical tasks and the influencing factors captured in Fig. 1 each play an important role in students' experiences. Tasks, however, do not *directly* influence student learning, nor does teaching. Rather, we adopt the perspective that the acts of a teacher and material resources in the form of curriculum and tasks influence the experiential environment of a student. This environment, in turn, mediates reasoning and learning (Carpenter et al., 1996; Remillard, 2005; Wittrock, 1987). Compatible with a constructivist perspective on teaching and learning (e.g., Simon, 1995; Steffe & D'Ambrosio, 1995), a mediated perspective on tasks and their implementation allows for viewing any particular learner's (or teacher's) experience in the classroom as idiosyncratic. The learner defines a task's influence.

The relationship between different phases of a mathematical task and individuals' experiences are captured by distinctions between *formal*, *intended*, *implemented*, and *experienced* curricula (Gehrke et al., 1992; Chap. 2; Kilpatrick, 2011; Remillard, 2005). Formal curriculum refers to the goals of an activity per its designers, while intended curriculum incorporates a teacher's intentions for the materials (Gehrke, Chap. 2 et al., 1992). The former aligns with Stein et al.'s first phase in Fig. 1, while the latter can be thought of as a transitional phase between Stein et al.'s first and second phase. This transitional phase involves a teacher building a personal image of a task and associated instructional goals. Implemented and experienced curricula—with the latter sometimes termed *enacted*, *attained*, or *realized* curriculum—each refer to classroom experience, but they differ in perspective. Implemented curriculum refers to the teacher's perspective (Kilpatrick, 2011), while experienced curriculum refers to a student's perspective (Gehrke et al., 1992; Chap. 2; Remillard, 2005). The former somewhat aligns with Stein et al.'s second phase in Fig. 1, and the latter with the third phase.

Informed by these collective perspectives on tasks and curriculum, we approach a researcher's account of a task and its implementation as the process of developing a hypothetical account that requires being selective and transparent with respect to focus. In this paper, we take a two-part focus. Firstly, we characterize mathematical tasks in terms of their mathematical goals and the ways in which students might engage with the tasks as presented. We incorporate both formal and intended aspects of curriculum by taking into account the participating teachers' stated goals when we develop our characterizations. Secondly, we draw on classroom data to describe the implementation of the mathematical tasks. We do this by developing hypothetical accounts of experienced curriculum as inferred from the ways in which the teacher and students set up and engage (or implement per Stein et al., 1996) with a task. We connect these two foci—that of design and that of implementation—by adopting a cognitively-based framework as described in the following section.

2.2. Abstraction as a theoretical framework

Abstraction is a term commonly used within both mathematics and mathematics education. In common mathematics lexicon, an abstraction refers to a generalized structure that is applicable to a class of real-world objects or instantiations of it, yet it is not dependent on any particular object or instantiation. The mathematical practice of conceiving and applying the same structure across different objects and contexts is sometimes referred to as decontextualization (Dreyfus, 2014). Pushing past this vague notion of decontextualization, mathematics educators have considered the process of abstraction itself and the type of generalized knowledge structures resulting from abstraction processes (Dreyfus, 2014). This has led to abstraction perspectives that vary in their epistemology and in what is considered to comprise an abstraction. For instance, the *Concrete-Representational-Abstract* (*CRA*) sequence (Fig. 2), which captures Bruner's (1966) three modes of representation, frames abstraction in terms of a representational context in tandem with decontextualization (Flores, 2009; Hinton & Flores, 2019; Miller et al., 2011; Peterson et al., 1988; Witzel et al., 2012). As another example of an abstraction perspective, Schoenfeld (1991) characterized the problem-solving process in terms of a translation between real-world situations and a formal system. He considered the latter as abstract structures to be *applied* to decontextualized aspects pulled from real-world situations (Fig. 3). In Schoenfeld's words, "formal systems in mathematics are not *about* anything. Formal systems consist of sets of symbols and rules for manipulating them" (1991, p. 311).

Traditional approaches to abstraction like those above foreground decontextualization and representational translations, while (implicitly or explicitly) treating formal or abstract structures as things to apply or represent symbolically when solving mathematical tasks or problems. A formal or abstract structure is entangled with its symbolic form and application. Other approaches to abstraction have attempted to depart from such entanglements, including that of Piaget.

Piaget's notion of abstraction has informed the work of mathematics educators across the K-16 spectrum. As a non-exhaustive list, notable contributions have occurred in fractional reasoning (Simon et al., 2016; Steffe & Olive, 2010), rate of change and accumulation (Thompson, 1994a, 1994b), function classes (Ellis et al., 2024), function (Dubinsky, 1991; Sfard, 1992), notation use (Tillema & Hackenberg, 2011), combinatorics (Antonides & Battista, 2022; Ellis, Lockwood, & Ozaltun-Celik, 2022), and limit concepts (Oehrtman, 2008). Researchers have built on Piaget's notion of abstraction to propose numerous knowledge constructs. Tzur and Simon (2004) proposed two stages of learning, *participatory* and *anticipatory*, to distinguish the nature of activity supported by processes of abstraction. Battista (2007) introduced a theory of levels of abstraction to explain spatial-geometric reasoning. Silverman and Thompson (2008) provided an account for *mathematical knowledge for teaching* rooted in repeated processes of abstraction that support a teacher in developing a knowledge base that has pedagogical power. Both Liang (2021) and Tallman (2015) built on this perspective to identify nuanced mechanisms involved in teachers' construction, abstraction, and enactment of such knowledge.

At the most general level, Piaget viewed abstraction as the mechanism or process of learning (Ellis et al., 2024; Piaget, 2001; Tallman & O'Bryan, 2024). Piaget viewed learning to involve aspects of assimilation, accommodation, equilibration, and perturbation. Assimilation is the process by which an individual conceives a present experience via their current conceptual structures (von Glasersfeld, 1995). A perturbation occurs when assimilation to extant conceptual structures results in an unexpected experience. Accommodation is then the elimination of a perturbation through a cognitive construction or reorganization, which establishes a cognitive state of equilibrium (Piaget, 2001; von Glasersfeld, 1995). With learning defined in this way, Piaget conceived abstraction as

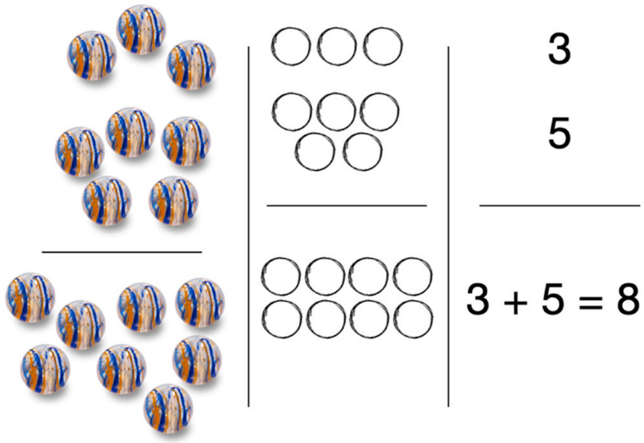


Fig. 2. The CRA approach for an addition problem involving 3 + 5, using marbles for the concrete, drawn circles for the representational, and numerical inscriptions for the abstract.

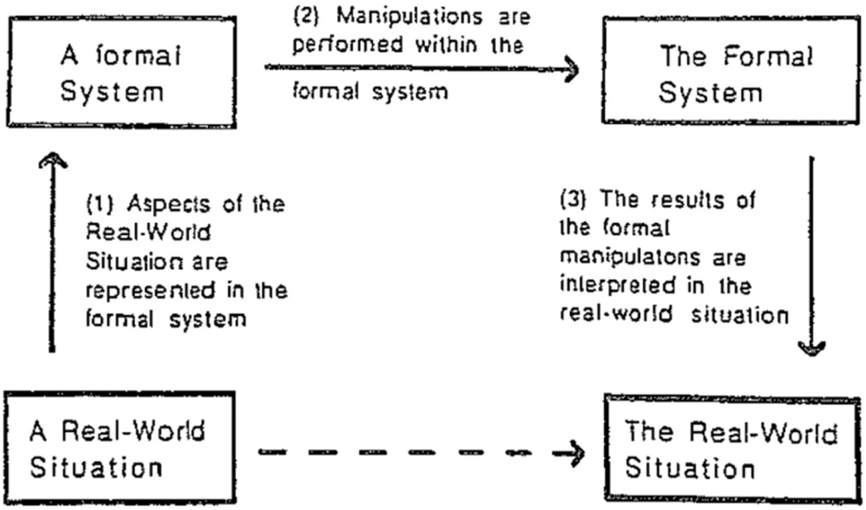


Fig. 3. The problem-solving process as a translation between real-world situations and a formal system composed of abstract structures and tools involving sets of symbols (Schoenfeld, 1991, p. 313).

a way to characterize and differentiate between various forms of stabilized knowledge structures, including how they might develop over time. In this paper, we choose to focus on Piaget's forms of abstraction in order to provide generalized characterizations of the learning processes and outcomes associated with tasks and implementations.

For the purpose of providing operational definitions of abstraction, we draw on two chapters (Ellis et al., 2024; Tallman & O'Bryan, 2024) that summarize Piaget's forms of abstraction and their application in mathematics education.[1] Piaget identified five varieties of abstraction: *empirical, pseudo-empirical, reflecting, reflected,* and *meta-reflection*.[2] We restrict our primary focus to pseudo-empirical abstraction and reflecting abstraction. We do so because empirical abstraction, reflected abstraction, and meta-reflection are less relevant to our focus. Empirical abstractions primarily concern observables and sensory-motor experience, and are thus more relevant to early developmental levels of mathematical reasoning (Ellis et al., 2024; Piaget, 2001). Reflected abstractions rest on a subject's consciousness of their ways of operating, and its form is thus more relevant to sequences of activities and reflection across those activities for the purpose of becoming conscious of generalized properties of various mental operations. Similarly, meta-reflection addresses the process of reflecting on one's own thinking and reflecting processes, and is also more relevant to sequences of activities and reflection across those activities. Each of these forms of abstraction can occur during a task, but pseudo-empirical abstraction

[1] We direct the reader to each chapter for a detailed synthesis of the history of Piaget's forms of abstraction including their use in mathematics education.

[2] We use reflective abstraction as a categorical reference to pseudo-empirical, reflecting, and reflected abstraction (Piaget, 2001).

and reflecting abstractions are more organic to the enactment of mathematical reasoning during initial learning experiences or isolated activity (Ellis et al., 2024; Moore, 2014; Piaget, 2001; Tallman & O'Bryan, 2024).

Speaking on pseudo-empirical abstraction, Piaget (1977) explained,

> When the object has been modified by the subject's actions and enriched by the properties drawn from their coordinations…the abstraction bearing upon these properties is called 'pseudo- empirical' because, while it concerns the object and its actual observable traits as in empirical abstraction, the facts it reveals concern, in reality, the products of the coordination of the subject's actions…" (p. 303).

To Piaget, a critical aspect of pseudo-empirical abstraction is that it foregrounds actions that require the presence of perceptual material. For instance, in order to count a collection of objects or determine the sum of two collections of objects, the individual might need the objects visually present or, at the least, rely on tactile taps or an imagined array to count and combine. Or, in the context of measurement, an individual may require a unit segment be present and useable to partition a given length and determine its measure.

Drawing on the work of Moore and colleagues (Liang & Moore, 2021; Moore, 2014; Moore et al., 2019), Ellis et al. (2024) argued for extending the construct of pseudo-empirical abstraction so that it does not hinge on the presence of "perceptual material" or "observables." They presented a framing of pseudo-empirical abstraction that foregrounds the *products of activity,* even if these products are purely cognitive and thus do not rely on the presence of figurative material. Ellis et al. (2024) illustrated that such an extension of pseudo-empirical abstraction is productive for developing viable models of students' mathematics at levels beyond that of elementary school. An example of pseudo-empirical abstraction includes students' construction of graphing meanings that foreground the sensorimotor or perceptual properties of drawn graphs (Moore, 2014, 2021; Moore et al., 2019). For instance, Moore et al. (2019) illustrated examples of students holding graphing meanings that *necessitated* the physical act of drawing graphs right-to-left, starting a graph on the y-axis, or exclusively associating slope properties with particular directional movements (Fig. 4). Such abstractions are consistent with Harel's (2001) notion of a result-pattern generalization, in which the knowledge structures a student constructs are emergent properties of results.

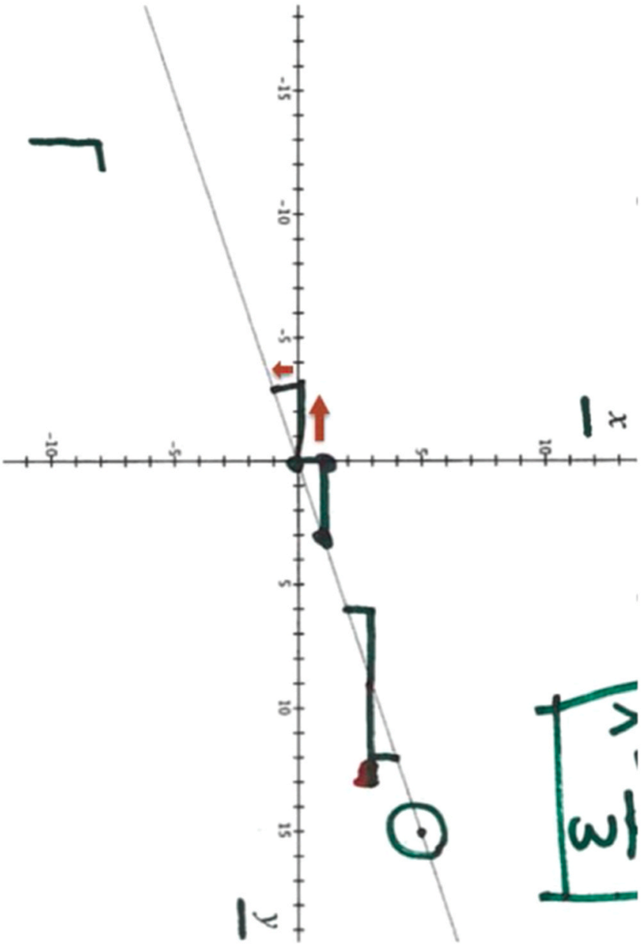


Fig. 4. A graph of $y = 3x$ that a student conceived as having a negative slope due to its right-to-left upward movement (Moore et al., 2019, p. 9).

Piaget's distinction between pseudo-empirical abstraction and reflecting abstraction rested on the extent to which perceptual material or observables are required. Ellis et al. (2024) noted that a broader interpretation of pseudo-empirical abstraction requires an alternative framing of reflecting abstraction that maintains its central tenet but does not center the availability or requirement of perceptual material. They addressed this issue by noting that a primary difference between these two forms of abstraction is that while the source material for pseudo-empirical abstractions is perceptual material or the result of actions, the source material for reflecting abstractions is the coordination of a subject's actions themselves. Reflecting abstractions involve differentiating an action from its effect so that the action itself can be projected to a level of representation and taken as an object of thought (Ellis et al., 2024; Tallman & O'Bryan, 2024; Thompson, 1994a). As we illustrate with our results, these differences in the source material for a student's abstractions have important implications for their learning.

To demonstrate the difference between pseudo-empirical and reflecting abstraction, we return to the graphing example above. Moore et al. (2019) shared a case (Annika) we consider to indicate reflecting abstraction. In response to being provided a linear graph but without a labeled coordinate system (Fig. 5a), Annika identified several label options that yield a viable graph of $y = 3x$ (Fig. 5b-c). As opposed to foregrounding perceptual material including sensorimotor actions, Annika's actions for slope involved not only coordinating two quantities' values or magnitudes to understand their covariation, but she also coordinated the relationships represented by each graph with each other so that she perceived each graph as equivalent in terms of the represented relationship (Carlson et al., 2002; Saldanha & Thompson, 1998). Her coordination of the quantities' covariation in tandem with multiple representations being produced by that covariation is consistent with reflecting abstraction. Her reasoning foregrounds the coordination of mental actions with results (including any perceptual or sensorimotor properties) so that the results are a consequence of those mental actions.[3] Her actions are also consistent with Harel's (2001) notion of a process-pattern generalization, in which the knowledge structures a student constructs are emergent properties of repeated processes.

Researchers often position reflecting abstractions as yielding more productive or expansive generalizations than pseudo-empirical abstractions. This can be the case due to reflecting abstractions involving both the results of actions and the actions themselves so that those actions are coordinated and projected to a higher-level of knowledge (Ellis, Lockwood & Tillema, & Moore, 2022; Ellis et al., 2024; Moore, 2014; Moore et al., in press; Piaget, 2001; von Glasersfeld, 1991). However, pseudo-empirical abstractions can also be productive and of developmental importance. Steffe's (Steffe & Olive, 2010) research on counting and fractional reasoning, for instance, includes numerous examples of children's productive pseudo-empirical abstraction. Steffe illustrated that these schemes represent critical and necessary developmental stages of learning. Pseudo-empirical abstractions can also provide the foundation for reflecting abstractions via providing insights and source material for further reflection that involves coordinating those abstractions with prior actions (Ellis et al., 2024; Piaget, 2001). Relatedly, the products of reflecting abstractions can become the source material for subsequent pseudo-empirical abstractions as a student identifies patterns in their own knowledge structures and abstractions. Our empirical examples below suggest that reflecting abstraction is the more powerful form of abstraction, but we underscore that pseudo-empirical abstractions are often important to a student's mathematical development (Ellis, Lockwood, & Ozaltun-Celik, 2022; Ellis et al., 2024).

3. Methods

3.1. Project setting

This present work is situated in a multi-year project investigating students' generalizing, including the ways in which teachers support generalizing in their teaching (Ellis, Lockwood & Tillema, & Moore, 2022; Ellis et al., 2017; Ellis, Waswa et al., 2024). Here, we focus on two in-service teacher participants, one at the high school level (grades 9–12) and one at the middle school level (grades 6–8). We chose the participating teachers by contacting nearby districts asking for teachers interested in participating in a study on supporting generalization in their classrooms. From the initial response of 6 teachers, we conducted classroom observations during a lesson of their choice. We observed the lessons to identify those teachers whose current classroom practices entailed student-centered activity and opportunities for student generalizations. These initial observations resulted in our narrowing the participant pool to three teachers. This paper focuses on Ms. N and Ms. R. The two teachers taught in different schools, with each school serving diverse student populations discussed in Ellis et al. (2024).

3.2. Data collection and analysis

In total, we observed two lessons in each teacher's classroom. The lessons in Ms. R's classroom spanned four and two days, respectively. The lessons in Ms. N's classroom spanned four and three days, respectively. Members of the research team recorded each lesson using two video cameras, one moving camera and one stationary camera. The moving camera was aimed at the teacher during whole-class discussion and small groups during group work time, and its audio recording was generated by a microphone worn by the teacher. The stationary camera focused on a group of three to four students chosen by the teacher. It captured their conversations,

[3] We note that another form of coordination could involve a student coordinating quantities of a phenomenon (e.g., rotating gears) to understand their covariation and subsequently coordinating aspects of a mathematical representation (e.g., the values of a graph or table) with those quantitative referents and the understood covariational relationship. Such cases, which foreground coordinating representational activity with the covariation of quantitative referents form the primary focus of this paper.

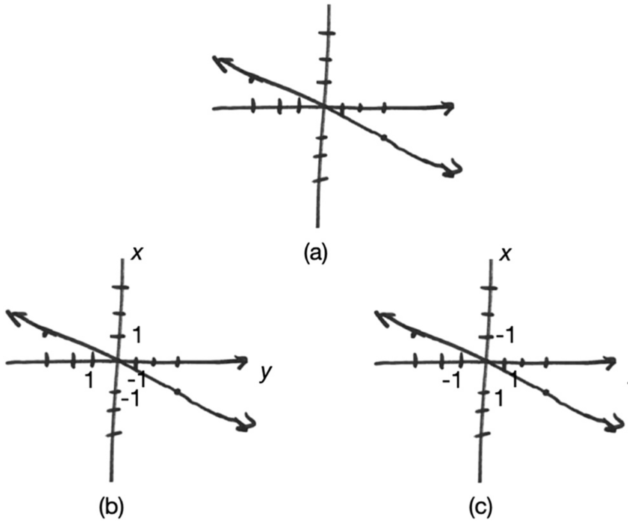


Fig. 5. (a) An unlabeled linear graph given to a student and (b-c) two labeled orientations that yield a viable graph of $y = 3x$.

written work, and gestures. We provide a summary of the lesson data collected in Table 1.

We analyzed the instructional tasks and classroom interactions using conceptual analysis (Thompson, 2008) with a guiding framework of the forms of abstraction identified above. Conceptual analysis involves answering the question, "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (von Glasersfeld, 1995, p. 78). With respect to analyzing instructional tasks or classroom interactions, conceptual analysis involves developing hypothetical accounts of realized or experienced curriculum. Such an approach rests on the stance that there is no objective curriculum or classroom interaction, and thus a researcher can only develop hypothetical accounts of those constructed meanings. Underscoring the cognitive orientation of our curricular analysis, this requires that the researcher persistently work to analyze data using models of student thinking they are aware of, either through their own research or the research of others. A researcher balances their first-order and second-order knowledge (Steffe & Thompson, 2000) for the purpose of explaining tasks materials and classroom interactions as potentially experienced by the students.

With respect to the instructional tasks, our conceptual analysis first involved generating "typical" solutions to the instructional tasks and accounts of the potential mental actions driving those solutions. We drew on our own research expertize in developing cognitive models of individuals' meanings for major middle and secondary grades topics. This expertize included our own studies (e.g., Ellis, 2011; Ellis & Grinstead, 2008; Fonger et al., 2020; Moore et al., 2022; Moore et al., 2019; Tasova, 2021), as well as our knowledge of cognitive models built by researchers outside of our research group (e.g., Byerley & Thompson, 2017; Carlson et al., 2002; Knuth, 2000; Thompson, 1993). With those accounts developed, we constructed hypothetical abstractions by considering the different ways in which students might reflect on their solutions and associated actions.

With respect to the instructional implementation of the analyzed materials, we first coded each classroom interaction in ways that captured the mathematical activity of the students and class. This round of coding is consistent with that reported in (Ellis et al., 2024), and it formed an important foundation for understanding the student activity and potential reasoning processes driving that activity (i.e., conceptual analysis). We subsequently coded for instances of student or teacher actions that suggested students engaged in potential pseudo-empirical or reflecting abstraction processes. We used a generic abstraction code to capture any other instance that might be relevant to developing hypothetical accounts of student abstractions, which reflects that the processes involved in abstraction can

Table 1
Lessons observed in Ms. R's and Ms. N's classrooms.

Teacher	Number of lessons	Length of each class	Class	Mathematical topics
Ms. R	2	80 min	Alg. 1	Linear equations and inequalities Writing & manipulating linear equations Graphing linear inequalities Solving systems of equations and inequalities Linear and quadratic growth* Multiple representations of quadratic functions
Ms. N	2	55 min	6th Grade Math	Plotting points in the coordinate plane Properties of the coordinate plane, scaling axes Horizontal and vertical distance between points Reflection of points across x- and y-axis Proportions, scale factors, equivalent ratios*

* Focus lessons for the present paper

occur over a time span greater than a single class or lesson. For instance, we coded instances in which an instructional move was made by the teacher that might influence the students' abstraction activity, such as a teacher presenting a solution in a way that privileges the process of generating a solution versus that of foregrounding the final state of a solution. We then used the results of this coding process to organize a narrative of the progress of the lesson as it relates to potential abstractions (whether realized or in progress).

In terms of executing the coding process, we first chose two tasks—one from each teacher participant—and each research team member independently coded these instructional tasks for abstraction. We then met as a group to collectively compare and reconcile our codes. This process enabled us to explore the viability of coding instructional tasks for potential abstractions while also working toward compatible meanings for the various forms of abstraction as they relate to the instructional tasks. We then had at least three research team members independently code the remaining instructional tasks, meeting as a subgroup to compare and reconcile their codes. They brought all remaining discrepancies to the entire research group for consideration, and a senior member of the research group reviewed all final codes. We repeated this process for coding the task implementation.

4. Results

We present two tasks that afford a representative range of potential abstractions. We first present an analysis of a task and its implementation from Ms. N's class, followed by an analysis of a task and its implementation from Ms. R's class. Each class introduced a key concept for their respective grade bands, with Ms. N exploring ratios and Ms. R addressing patterns of quadratic (and linear) growth. For the task analyses, we organize our results using the two forms of abstraction. For the classroom implementation analyses, we provide a narrative synthesizing the evolution of the lesson while highlighting salient moments with respect to abstraction forms.

4.1. Ms. N, gears, and ratios

4.1.1. Task analysis

Ms. N's instructional task used the context of pairs of different sized gears to explore ideas of ratio and proportion, and it was an adaptation from Ellis (2007a) (2007b) (2007c). Students were asked to reason about gear pairs in order to determine and explore equivalent ratios. Ms. N first prompted students to think about the relationship between the number of big gear rotations and the number of small gear rotations (Fig. 6). To support their exploration, students had two physical gears with eight and 12 teeth (Fig. 7), respectively, and were tasked with finding a way to record the number of rotations that each gear made.

Students then engaged in a series of tasks that explored rotation pairs, which first focused on equivalent ratios with only whole number rotations and then moved on to generalize to fractional rotations. For example, students had to determine whether given pairs of rotations were "correct" (see Fig. 8). By correct, the teacher meant that each of the rotation pairs in a table came from the same pair of gears. The tasks also asked students to imagine hypothetical gear pairings with particular rotation relationships (see Fig. 9). Ms. N intended that the students leverage equivalent ratios and related ideas (e.g., scaling), with the physical gears providing concrete materials to use as necessary.

There are various ways that a student might approach the rotation pair activities, with one being to simply take the initial rotation pair as a composed unit (Lamon, 1994) and iterate it to see whether the iterations would eventually yield all of the rotation pairs in the table (Fig. 10a). Another related strategy would be to determine if the same scale factor determines the number of rotations of the small gear and big gear based on the initial pair of rotations (Fig. 10b). Another solution could entail determining if the number of big gear rotations is a constant multiple of the small gear rotations.

With respect to a task like the one in Fig. 9, a student might explicitly attend to each gear's number of teeth. The student might imagine that as the two gears turn, their teeth meet up in a zipper fashion and thus both gears turn through the same number of teeth regardless of how many full rotations they make. This observation, when combined with understanding that the number of teeth rotated is a proxy for number of rotations, could lead the student to conclude that for the big gear to turn twice as many turns for a rotation of the small gear, then the small gear will need twice as many teeth. More generally, to make the 12-tooth gear rotate *N* times as many turns as it did originally, the number of teeth in the second gear must increase by a factor of *N*.

Alternatively, a solution to a task like that in Fig. 9 could involve using the relationship between rotations of the original two gears. For example, a student could imagine that every time the eight-tooth gear completes a full rotation, it will have rotated through eight teeth. Thus, the 12-tooth gear will also have rotated through eight teeth, which is $\frac{8}{12}$ or $\frac{2}{3}$ of a full rotation. Because the new gear pairing

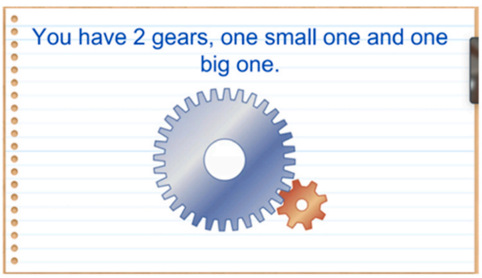


Fig. 6. The opening context for the task.

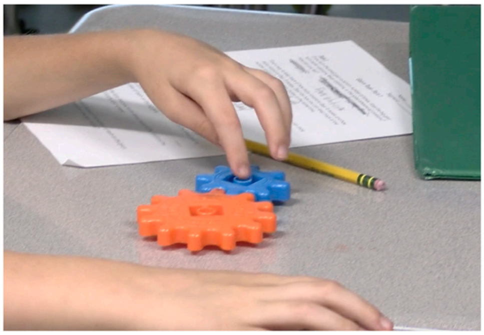


Fig. 7. The physical gears Ms. N provided to the students.

Small	Big
6	4
9	6
12	8
15	10
18	12
21	14
24	16

(a)

Small	Big
4	3
8	6
16	12
24	18
48	36
2	1.5

(b)

Fig. 8. Two different tasks (a-b) to determine whether the entries are "correct".

Say we're working with the small gear (8 teeth) and the medium gear (12 teeth).

1. If you could replace the small gear with a different gear that would make the big gear turn twice as many turns, how many teeth would that different gear have?
2. What if you wanted to replace the small gear with a different gear to make the big gear turn half as many times instead of twice as many? How many teeth would that different gear have?
3. What if we wanted the big gear to turn **3** times as many turns? **10** times as many? **N** number of times?

Fig. 9. An example task prompting students to extend their reasoning to hypothetical gears.

must be such that when the small gear's replacement rotates one time, the 12-tooth gear rotates $2 * \frac{2}{3}$, or $\frac{4}{3}$ of a full rotation, which is $\frac{4}{3} * 12$, or 16 teeth. Thus, the new gear must have 16 teeth. Similarly, for the second question a student could imagine that in order to have the 12-tooth gear turn half as many times, then it must rotate $\frac{1}{2} * \frac{2}{3}$, or $\frac{1}{3}$ of a rotation and thus through $\frac{1}{3} * 12$, or 4 teeth. The second gear must have four teeth. A student could then generalize that if the 12-tooth gear turns N times as many turns as it did to start, then it will turn $N * \frac{2}{3}$ full rotations and therefore go through $N * \frac{2}{3} * 12$ teeth.

Pseudo-Empirical Abstraction. Because these tasks involved the use of physical gears to explore the relationships between gears' rotations, with the initial task prompting students to explore using the gears to determine paired rotations, there is ample opportunity for students to develop pseudo-empirical abstractions. For example, when students reason about the table in Fig. 8a, they can physically rotate their small gear three times and notice that their large gear rotates twice, and that if they rotate their small gear another three times, the large gear will again rotate an additional two times, giving six and four total rotations respectively. As the student checks each entry in the table, they can either carry out or imagine the small gear rotating three times and the large gear rotating twice each time they move down a row. This reasoning would involve pseudo-empirical abstraction because the student is reliant on the

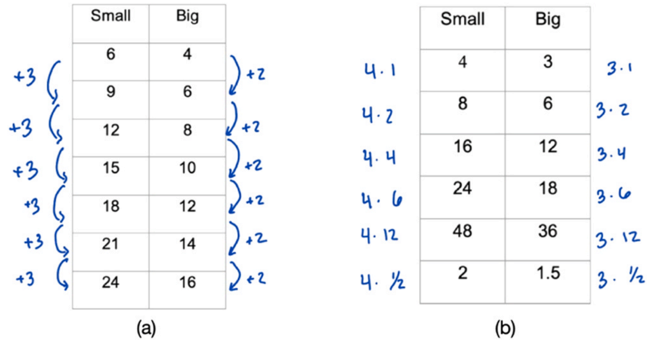


Fig. 10. Hypothetical examples of student work (a-b) to determine whether entries from each of the tables could be made from the same set of gears.

perceptual material or their mental image of that material in order to solve the task.

The tables themselves, in tandem with the produced pairs from the physical gears, also offer opportunity for the development of pseudo-empirical abstractions, as students can notice patterns in the table values and use these identified patterns to confirm that each pair of values satisfies that pattern. Using the table in Fig. 8a, a student might notice that each row is obtained by simply adding three to the value in the lefthand column in the previous row, and two to the value on the right. Although this reasoning does not require the rotation of physical gears, it is a pseudo-empirical abstraction because the student does rely on the table (i.e., the results of activity), which becomes the figurative material that is the basis for the abstraction. Without the table, the student would be unable to determine possible rotation pairs because their abstraction is based on extending the pattern from the produced pairs of values. This strategy works for that particular table, but it could lead students to assume that a rotation pair would not be from the same set of gears if it did not continue the iteration pattern (e.g., a table skipping from 12 and eight rotations to 21 and 14 rotations as in Fig. 11). Similarly, it could lead students to assume that two tables stem from different paired gears due to the patterns having different values. Moving to the task in Fig. 9, a student might extend their previously abstracted patterns to conclude that any change to the small gear yields the same change in the big gear (e.g., if the big gear turns two times as much, then the small gear must have two times the number of teeth), thus preserving the properties of the numerical patterns (e.g., if I double a number with the big gear, I double a number with the small gear).

Reflecting Abstraction. Returning to the tables in Fig. 8, a student could instead develop a reflecting abstraction that no longer relies on using or imagining their gears, or extends beyond tabular numerical patterns so that identified patterns are coordinated with invariant properties of the physical gears. With the table in Fig. 8a, a student might recognize that every time the small gear rotates three times the large gear will rotate twice, and thus for any additional sets of three rotations of the small gear, there will be an equal number of sets of two rotations of the large gear. In this case, the student can think about this relationship without having to imagine the gears rotating three and two times, and at the same time maintain a persistent realization of the invariant relationship between the gears. This same abstraction can be used to reason about the gear pairing in Fig. 8b as well, with a student recognizing that every time the small gear rotates four times, the large gear will rotate three times. Therefore, if the small gear rotates $4 * n$ times, the large gear will rotate $3 * n$ times. This would require a student to coordinate the rotations of the large gear and the rotations of the small gear and recognize that as one gear rotates, the other gear will rotate as well according to this invariant relationship. Because this abstraction situates patterns so that they entail the coordination of actions that would produce the patterns, as opposed to merely observing the numerical patterns themselves, it could also lead students to conceive that a rotation pair is from the same set of gears in the presence of those tables in Fig. 11. Here, the student could anticipate that the invariant relationship between the gears is reflected in any numerical pattern that preserves that invariant relationship.

Moving to the task in Fig. 9, a reflecting abstraction would involve a student coordinating the rotations of the 12-tooth gear and the

Small	Big
6	4
9	6
12	8
21	14
24	16

(a)

Small	Big
30	20
36	24
42	28
48	32
54	36

(b)

Fig. 11. A table (a) that is not ordered by incremental increases in the small gear, and a table (b) that is from the same pair, yet would provide a different numerical pattern.

hypothetical gears, but without the need to actually imagine rotating (or physically rotating) the 12-tooth gear a given number of times and then counting the teeth rotated through. Furthermore, the student would come to anticipate that the invariant relationship between the two gears rests on the relative relationship between the number of teeth forming each gear, and any numerical pattern between number of rotations stems from that relationship; the student coordinates the relationship between the gears so that numerical patterns are a product of that relationship. For instance, a student could realize that if a gear A has x number of teeth and gear B has y number of teeth, then gear A rotates y/x times for each single rotation of gear B due to the gears rotating the same number of teeth regardless of rotation amount. They could then use this relationship to solve problems with any gear pairing without needing to use manipulatives or even to imagine the rotations occurring (e.g., producing twice as much turn in gear A requires going through twice the number of teeth, thus requiring B to double in number of teeth).

4.1.2. Implementation analysis

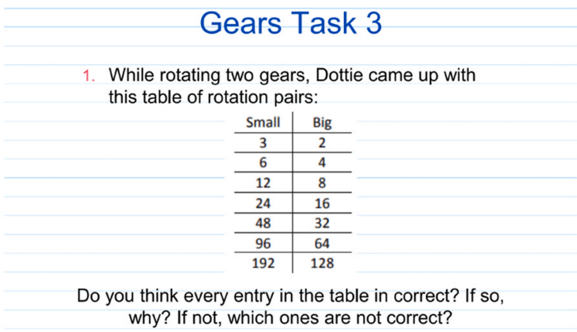
The students' actions during the tasks suggested both potential pseudo-empirical and reflecting abstractions. Regarding pseudo-empirical abstraction, there were several instances in which the students used tables of values to construct and use numerical patterns, but without evidence that these patterns were coordinated with properties of the rotating gears. Notably, there were instances in which students experienced perturbations from their table use, and it was in their reengagement with the gear contexts that their actions were suggestive of reflecting abstractions.

As an example of pseudo-empirical abstraction, consider the first activity day. Ms. N presented a table (Fig. 12) and asked the students to decide whether the rotation pairs came from a single pair of gears. One student, Jackson, first suggested a method of executing the spins, and then Kyle noticed a doubling pattern as he moved down the rows (Excerpt 1).

Excerpt 1.

Speaker	Transcript
Jackson:	I don't think I can do 190 spins.
Ms. N:	So I would, I would recommend not trying that. I agree. How could you figure it out without spinning it that many times?
Jackson:	[*To Kyle*] You got 192? No
Ms. N:	[*To Kyle*] How did you fill that? Did you spin it that many times or what else did you do to get those numbers on there?
Kyle:	Multiplied
Ms. N:	Multiplied what?
Kyle:	By two.
Ms. N:	Moving down your table. You multiplied what side of the table by two?
Kyle:	Three times two is six.
Ms. N:	Okay.
Kyle:	Six times two is twelve
Ms. N:	Okay.
Kyle:	twelve times two is 24
Ms. N:	Great.
Kyle:	24 times two is 48.
Ms. N:	Yeah
Kyle:	48 times two is 96.
Ms. N:	So so far, you totally agree with Dottie's table?
Kyle:	Yeah.
Ms. N:	Awesome. What do you do to your big side?

Throughout this interaction, Kyle attended to one side of the table at a time, focusing first on the column of the small gear's rotations, and then, once prompted, addressing the big gear's rotations in a similar way. He identified that the subsequent row of the table is the previous row multiplied by two, which he used to conclude that the table of values was from a single set of gears. Thus, Kyle's actions are at least suggestive of a pseudo-empirical abstraction involving constructing and using a numerical pattern using the table of values. We do not have evidence in this interaction that Kyle was coordinating his tabular activity with properties of the gears (e.g., every time the number of rotations of the small gear doubles, the number of rotations of the big gear must also double), instead



Gears Task 3

1. While rotating two gears, Dottie came up with this table of rotation pairs:

Small	Big
3	2
6	4
12	8
24	16
48	32
96	64
192	128

Do you think every entry in the table in correct? If so, why? If not, which ones are not correct?

Fig. 12. A task from the first day of the activity.

leaving to possibility that his reasoning and inferences were restricted to perceived properties of the table. The likelihood of this possibility is strengthened by the fact that his attention focused on one side of the table at a time, which is a contraindication that he was coordinating pattern seeking with properties of relationships between the two gears.

As a related example of pseudo-empirical abstraction, at other times students compared entries across rows in tables. For instance, a student reasoned about a rotation pairs table (Fig. 13) to identify that he could divide the number of rotations of the small gear by three and multiply the resulting number by two to determine the number of big gear rotations. Through identifying this pattern, he concluded that the paired values were correct because each entry maintained the pattern. Like above, when discussing his strategy, the student maintained an exclusive focus on the table and he did not provide evidence of coordinating constructed patterns with the physical properties of the gears. Thus, it is possible that the constructed pattern and the fact it held throughout the table was the foundation for his conclusion without an intrinsic connection to the fact that one gear must always rotate some constant number of times as the other gear in a paired gears scenario.

The examples above illustrate that a difficulty in analyzing classroom interactions for potential abstractions is that those interactions can provide limited evidence for analysis due to the substance of questioning and dialog. This is particularly true with respect to evidence for reflecting abstractions. In both cases above, the interaction proceeded without the student or teacher drawing explicit attention to the gear situation as it relates to their tabular activity. Thus, we did not have evidence of their tabular activity being coordinated with that of the physical gears and potential invariant properties of them. But, absence of evidence is not evidence of absence. We can thus claim evidence for pseudo-empirical abstractions, but we cannot claim evidence for the absence of reflecting abstractions.

As a comparison, consider Dan's solution to a third task (Fig. 14). Dan used analogous reasoning to Kyle with the task in Fig. 12 to conclude that the table was not produced by a viable set of gears (Excerpt 2).

Excerpt 2.

Speaker	Transcript
Ms. N:	Which ones do you not think are correct?
Dan:	24 and 18.
Ms. N:	Ok, why not?
Dan:	It goes from doubling, and then it just switches to this.

In this case, the student attempted to extend a doubling pattern from the (16, 12) pair to the (24, 18) pair. Because the latter pair was not the result of doubling the prior pair, the student concluded that the latter pair could not be produced by the same pair of gears. The student's reasoning is suggestive of a pseudo-empirical abstraction similar to those above, and their rejection of the (24,18) pair provides a contraindication of a reflecting abstraction. A reflecting abstraction would involve coordinating their patterns with an invariant relationship in the actual gears' rotations.

Ms. N next pushed Dan to reason using the gears rather than the table (Excerpt 3).

Excerpt 3.

Speaker	Transcript
Ms. N:	What if I didn't want to double it? What if I just wanted to do four rotations on the small gear? Is that not allowed? Do you have to always double it each time, or do you think that you could do 24 small rotations and the big gear would still move?
Dan:	I think we could do 24.
Ms. N:	So how do you know if it would actually match up at 18 if I did that?

As further evidence that Dan had engaged in pseudo-empirical abstraction, Ms. N's proposal to the student led to a perturbation that stemmed from their tabular activity not being connected to their actions with the gears. Dan was unable to resolve their perturbation, but Ms. N's encouragement to return to the context of the gears had the potential to support him in more powerful reasoning. We inferred an example of this type of reasoning in an interaction from the first day of the activities, when two students working together explained how they reasoned about the table in Fig. 12. Ms. N had asked what students thought about the big gear rotating 16 times when the small gear had rotated 24 times. Tom said that the 16 rotations would be correct, and explained his

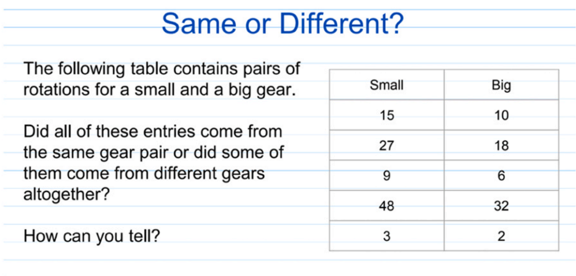


Fig. 13. A potential rotation pairs task.

Fig. 14. A second potential rotation pairs task.

reasoning (Excerpt 4).

Excerpt 4.

Speaker	Transcript
Tom:	… I just look at this number [*the number of rotations of the big gear*] if it is telling me how many one and a halves it is supposed to have.
Ms. N:	Cool
Tom:	Like here one and a half and one and a half would be three [*referring to the first row of the table*].
Ms. N:	Oh, yeah.
Tom:	And here four would be six [*referring to the second row*].
Ms. N:	How do you know that you can think about in terms of how many one and a halfs?
Tom:	I just did.
Ms. N:	Well, what did we write on this side? How does that relate to this?
Julia:	Because this one is gone one time, while this one is half way there.

This interaction suggests that Tom and Julia had abstracted that each time the big gear rotated once, the small gear rotated one full rotation, plus an additional half of a rotation. This meant that however many times the big gear rotated, the small gear would have rotated 1.5 times that number of rotations. In contrast to the previously discussed examples, this excerpt suggests that these students had engaged in reflecting abstraction. While their discussion was situated within the context of the gears, they were not reliant on physically or mentally manipulating the materials to determine whether the entries in the table were correct. They instead had extracted the invariant relationship between the rotations of the two gears and then coordinated that relationship with the construction of the table and numerical patterns in values. Tom and Julia engaged in pattern recognition, but maintained a recognition that the invariant properties of the gears were at the root of any recognized patterns.

4.2. Ms. R, patterns of growth, and formulas

4.2.1. Task analysis

Ms. R's instructional task explored quadratic growth compared to linear growth via a sequence of discretely growing shapes (Fig. 15). The first two shapes introduced a logo of varying sizes (Fig. 15a-b), the second two shapes introduced posters of different sizes (Fig. 15c-d), and the third two shapes introduced sails of different sizes (Fig. 15e-f). A primary goal of the lesson was to identify that quadratic relationships are such that equal increases in the values of the independent quantity (e.g., sail size) result in the values of the dependent quantity (e.g., sail area) increasing by constantly increasing amounts (i.e., a constant second difference). In service of this goal, each task prompted the students to determine the area for the 10th size and generate a formula for the area of the n^{th} size. For length purposes, we situate our discussion within the sail shapes (Fig. 15e and Fig. 15f).

We discuss each form of abstraction using a typical student solution presented in Fig. 16. The solution involves a student generating a table of size and area values for growing figure sizes 1–4. With those values, a predominant student action involves identifying both first and second differences in the area values, ultimately identifying constant first (Fig. 15e) or second differences (Fig. 15f). The student could then engage in a variety of actions to determine the sail areas for size 10, as well as a formula relating the sail size to sail area for each case. With respect to the sail areas for size 10, common student actions involve using the patterns of first differences in each case to identify the additional area needed from size 4 to size 10 (6 *1.5 for Figs. 15e or 18 +22 +26 +30 +34 +38 for Fig. 15f). Or, and in tandem with determining a formula, a student might identify a pattern between size and area values. With respect to the values in Fig. 16 (left), a student might conceive the area value is always 1.5 times as large as the size value, yielding $A = 1.5s$. With respect to the values in Fig. 16 (right), a student might conceive the area value is always 2 times as large as the squared size value, yielding $A = 2s^2$.[4] After working a series of activities like those presented in Fig. 15, and producing solutions like those in Fig. 16, a student might abstract the constant first (e.g., Fig. 15a, c, and e) or second (e.g., Fig. 15b, d, and f) difference patterns in a quantity's

[4] The actions involved in constructing these conceptions also involve potential abstractions. For the purposes of this paper, we focus on abstractions related to the more general goal of the lesson, rather than abstractions that are related to a singular context or task. These localized abstractions can contribute to the abstractions that span actions occurring over a sequence of contexts and a lesson.

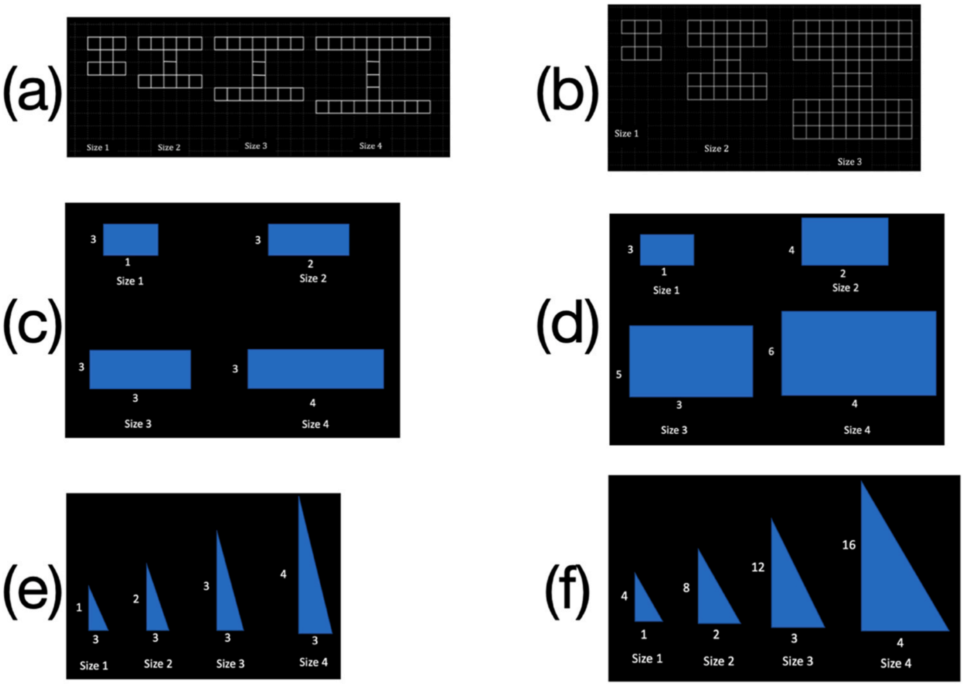


Fig. 15. The sequence a.-f. of discretely growing shapes implemented by Ms. R.

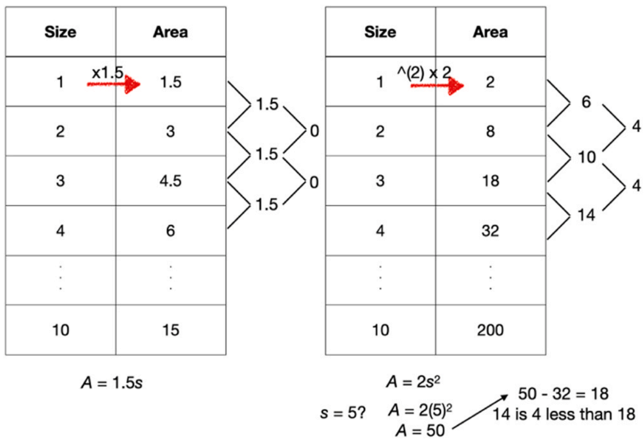


Fig. 16. A potential example of student work consistent with Ms. R's intentions for Fig. 15e (left) and Fig. 15f (right).

values in association with linear and quadratic relationships. We discuss potential abstractions with respect to those patterns.

Pseudo-Empirical Abstraction. In the case of the hypothetical solution presented above (Fig. 16), the products of activity include a table of calculated values and difference amounts. In the case of properties of first or second differences, an example of a pseudo-empirical abstraction is a student forming an association strictly based on noticing that constant first differences and constant second differences are accompanied by a linear formula and quadratic formula, respectively. In such a case, the abstraction consists of observing and remembering the co-occurrences of constant first or second differences and features of a formula. The actions that produced the table of values and formula are inconsequential to the abstraction except in that they yielded results that become the source material for the student's abstraction.

Based on our experiences with students, such an abstraction often results in the student associating a quadratic formula with constant second differences in the values of the dependent quantity regardless of how the other quantity's values are ordered in a table. For example, students may only attend to differences in the *y*-values of a table without coordinating that growth with the corresponding growth in *x*-values (as seen in Ellis, 2011; Fonger et al., 2020; Waswa, 2023). Because the abstracted association is stripped of the actions that produced the table of values, the mathematical properties of those actions (e.g., constant first differences) are not intrinsically tied to the abstraction. It follows that the association is not understood as a consequence of those mathematical properties.

As we illustrate in the following section, a different form of association entails a structure of mental actions that involves coordinating the co-occurrence of first and second difference properties with growth of the sail.

Reflecting Abstraction. In the case of the hypothetical solution presented above (Fig. 16), a reflecting abstraction that foregrounds the coordination of actions and their results would involve a student reflecting upon the quantitative referents of their tabular activity. Whereas the pseudo-empirical abstraction described above foregrounds using the tabular area values for calculational purposes, a reflecting abstraction involves being persistently aware that first differences represent the amount by which a quantity's values increases (or decreases) and second differences represent the amount by which the increase or decrease of a quantity increases (or decreases). Central to reflecting abstraction is coordinating these changes in quantities' values with their quantitative referents as shown in Fig. 17 so that abstracted patterns are coordinated with quantitative operations. Furthermore, central to reflecting abstraction is the awareness that the constant "+ 1" increases in size and side dimension(s) are necessarily tied to the area increasing by constant amounts (Fig. 17, top) and area increasing by constantly increasing amounts (Fig. 17, bottom); variation in one quantity occurs simultaneously with variation in the other quantity, and if size (or, technically, side) change is not constant, then those same first and second difference patterns do not hold. Importantly, such an abstraction involves conceiving numerical patterns as rates of growth so that those patterns are coordinated with their quantitative referents (see Fonger et al., 2020 for a detailed learning trajectory for quadratic growth).

4.2.2. Implementation analysis

Similar to Ms. N's implementation, Ms. R's implementation included contextual settings (e.g., pictures of discretely growing figures) and tables of values representing different quantities such as length and height. As Ms. R implemented the tasks, the students' actions, combined with how she dictated the direction of the classroom conversation, were suggestive of pseudo-empirical abstractions. For instance, students used tables to calculate first and second differences in triangle areas as the size of the triangles increased in a discrete manner. If the students determined there was a constant first difference, they then wrote a linear function. Similarly, if students determined there was a constant second difference, they then wrote a quadratic function. In each case, numerical patterns and writing formulas formed the primary classroom focus.

Due to the consistency of activity in her classroom, we describe one example of classroom activity in order to illustrate an image of the interactions and focus in Ms. R's classroom. Ms. R engaged students in a task in which they explored patterns of growth as both the width and the height of the sail grew proportionately in the ratio 1:4 (see Fig. 18). Ms. R asked students to determine "the area of the sail of size 10," and then write an "equation" to represent the growing area of the sails. Fig. 19 captures Ms. R's board annotation as she facilitated the whole class discussion. An excerpt of the whole class discussion is presented in Excerpt 5.

Excerpt 5.

Speaker	Transcript
Ms. R:	So the hard challenge is to make an equation [*writes y =* , *and sketches a table of size and area*].
Kate:	Well, it's squared cause it's quadratic.
Ms. R:	Alright. So, so make an equation and then we'll be done. So size, I'm gonna write my table to help me out. So, [*referring to size and area pairs*] one matches with two, two matches with eight, three matches with the 18. So four, what will four be? Did y'all figure that one out?

(*continued on next page*)

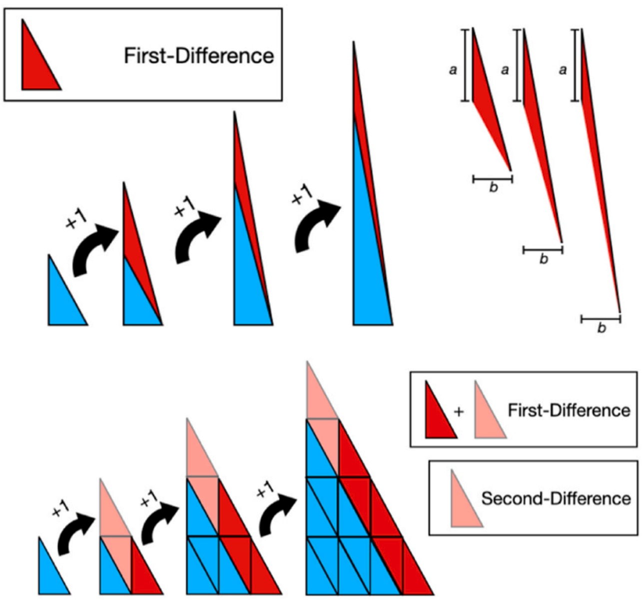


Fig. 17. Conceiving first and second differences quantitatively.

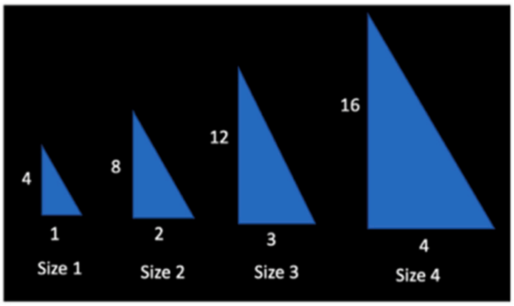


Fig. 18. Two-dimensional growing triangles.

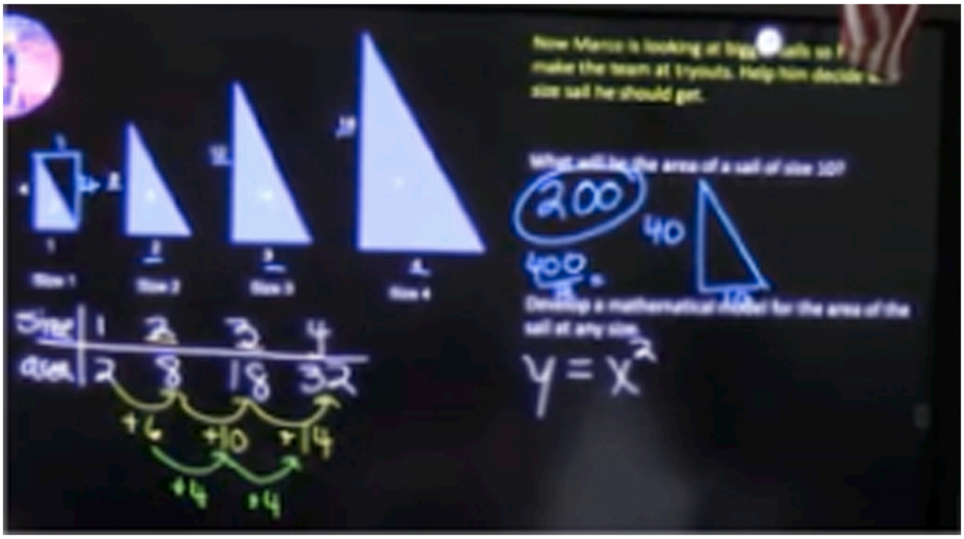


Fig. 19. Ms. R's annotations on the board for the two-dimensional growth case.

(*continued*)

Matt:	32, or what was it?
Kate:	Yeah, 32.
Ms. R:	32?
Kate:	Yeah, I knew that one for sure.
Ms. R:	Okay. You think? Alright, so if I look at my first difference [*writes the first differences in area below the area*], I'm adding six, adding 10, then what am I adding right there?
Student:	14
Ms. R:	14. Alright, then my second difference [*writes the second differences in area below the first differences*].
Kate:	Is plus 4.
Ms. R:	Plus four, plus four. Okay, so Kate said it's quadratic [*referring to Kate's earlier claim*]. So what do I need when it's quadratic?
Jake:	[*Cross talk*] You need a squared. You need the square.
Other student:	[*Cross talk*] Square.
Ms. R:	Yep. We need a square [writes $y = x^2$*; see Figure 19*]. So we know there's gonna be an x^2 in here. I don't know if there's a space in the front or not. So how can I manipulate these inputs [*referring to size*] to spit out those outputs [*referring to area*], when I know I need a squared? So that's, that's what you're trying to do. Do it in your groups, see if you can figure it out and then we'll be done.

This interaction is emblematic of Ms. R's classroom interactions throughout the lesson. Firstly, Ms. R and her students focused on calculating first and second differences using a table of values. They then used those values to identify patterns. As can be inferred from Kate's contributions, Ms. R intended that the students associate those constructed numerical patterns with linear and quadratic function forms. Secondly, Ms. R encouraged using trial and error to construct formulas representing their produced table of values, and by extension the growing sails. In the above interaction, she specifically encouraged students to "manipulate these inputs" so that the formula produced the corresponding surface area. Thirdly, the interaction captures that during the lesson, Ms. R moved the students forward after they obtained a correct formula that served as a record of the produced values and nothing more.

As it relates to abstraction, the interactions that occurred during Ms. R's classroom provided evidence of pseudo-empirical abstractions. Namely, her lesson and classroom interaction focused nearly exclusively on producing tables of values, performing calculations using those values, determining formulas sometimes through trial and error, and forming associations between patterns in the values and formula forms. Stated more generally, her questioning and guidance, as well as her students' contributions, were focused on producing and calculating values, executing actions for the purpose of determining formulas, drawing associations between the results of activity, and then basing progress on obtaining a correct solution. Consistent with our analysis of Ms. N's classroom interactions, we are compelled to note the difficulty in analyzing classroom interactions for potential abstractions, especially as it relates to evidence for reflecting abstractions. As suggested in our task analysis, evidence of reflecting abstractions might include explicit attention to the growth situations as they relate to the tabular activity and noted associations. Whereas Ms. R's classroom often moved forward when correct values and formulas were obtained, evidence for reflecting abstraction would involve coordinating their produced values with the growth of the sails including how the growth of the sails result in either constant rate of growth or constantly changing rates of growth (see Fig. 17). Based on the students' observable behaviors and products, we did not have evidence of their coordinating their tabular or formula activity with those operations involved in constructing quantitative growth.

5. Discussion

The purpose of the present work is to explore the question, "In what ways does abstraction provide a viable tool to characterize potential learning as it relates to instructional tasks and their implementation?" We adopted Piagetian forms of abstraction to analyze two teachers' instructional tasks and subsequent implementation. Although we are not aware of studies that have used the aforementioned forms of abstraction to develop hypothetical accounts of student activity in the context of teachers' instructional tasks, mathematics education researchers have been sensitive to the role of abstraction in instructional design. For example, Oehrtman (2008) provided a more general description of how Piaget's notion of abstraction can inform a layered sequence of activities so that students have an opportunity to reflect upon and identify common structures in their actions across a variety of contexts. We find it important to include a complementary focus on extending a construct that emerged from modeling cognition in order to characterize practicing teachers' instructional tasks, as those materials have a profound influence on students' educational experiences.

At its most general level, our work illustrates that adopting Piaget's forms of abstraction enables analyzing instructional tasks and their implementation in ways sensitive to a theory of learning and situated within mental actions. By adopting a grounding sensitive to the coordination of mental actions, it enables looking past surface level similarities and differences in student activity and solution products to provide differentiated accounts of student reasoning that generates those products. These differentiated accounts are not strictly alternative explanations of reasoning that might occur during engagement in and implementation of a task. Rather, these differentiated accounts can include an eye toward development via articulating pseudo-empirical abstractions, reflecting abstractions, and how pseudo-empirical abstractions might provide grounding for subsequent reflecting abstractions via designed learning environments. We return to this point throughout this discussion section and the subsequent future work section.

Reflecting on our analysis across the instructional tasks, the abstraction framing provides a guiding lens to produce differentiated accounts of anticipated knowledge development. In each of the two tasks we analyzed, we were able to use the different forms of abstraction to generate generalizations that students might construct as they reflect on their solutions and activity. Echoing Harel's (2001) distinction between result-pattern and process-pattern generalizations, distinguishing pseudo-empirical from reflecting abstraction helps us consider how a student might reflect on their solution activity. Analyzing instructional tasks with pseudo-empirical abstraction in mind enabled considering the different ways students might reflect on the results of their activity, such as identifying numerical patterns in produced tables or constructing associations between produced objects such as formulas and diagrams. Analyzing instructional tasks with reflecting abstraction in mind enabled identifying various ways in which students might coordinate the results of their activity with the actions driving their solution activity, such as coordinating patterns in a table of values with constructed invariant relationships in a context. The forms of abstraction thus provide guiding constructs to consider not only the cognitive actions that drive solution activity, but also the nature of reflection on that activity.

With respect to each implementation, the two forms of abstraction enabled analyzing classroom participant interactions including their produced artifacts in order to hypothesize their constructed abstractions or generalizations. This use of abstraction to analyze classrooms thus offers a framing for what Lobato et al. (2003) termed *focusing phenomena*. Focusing phenomena are the "features of the classroom environment that regularly direct attention to certain mathematical properties or patterns" (Lobato et al., 2003, p. 2). In this paper, the forms of abstraction enabled characterizing the cognitive activity potentially behind acts of identifying mathematical properties or patterns in the classroom, which yields two differentiated ways to frame focusing phenomena in a classroom. On one hand, the focusing phenomena occurring in a classroom might direct students toward reflecting on the results of their activity, such as emphasizing produced solutions or artifacts and properties thereof. On the other hand, the focusing phenomena occurring in a classroom might direct students toward a persistent focus on coordinating produced artifacts with the actions driving the production of those artifacts. With respect to the former, Ms. R's classroom sustained a focus on using produced tables and formulas to form associations between them. With respect to the latter, Ms. N's classroom interactions also included a focus on the ways in which observed patterns in produced tables were reflecting of properties of the physical gears and the students' actions with them.

Comparing the task analysis to the implementation analysis, the limitations that occur in analyzing classroom implementation are notable as compared to analyzing instructional tasks. With respect to the instructional tasks, the forms of abstraction proved useful and viable in several ways. Firstly, our analysis illustrates that the forms of abstraction enable operationalizing research-based models of student reasoning to articulate differentiated forms of anticipated student engagement and learning during instructional tasks. Our use of research-based models of student reasoning and associated theoretical constructs thus provides one response to calls for incorporating research and theory on student thinking in ways that are more attuned to the teaching and learning of mathematics as it occurs in the classroom (e.g., Ellis, 2022; Simon, 1995; Steffe & D'Ambrosio, 1995; Thompson, 2013). Secondly, our task analysis proved viable independent of the implementation analysis. We were intentional to analyze the tasks using our expertize and relevant research before analyzing implementation data so as to not have our task analysis influenced by implementation. For this reason, we anticipated a need to revise, perhaps significantly, our task analysis based on implementation analysis. This was not the case, as there was compatibility between the results of our task analysis and that of the implementation analysis. This outcome contributes to the viability of using forms of abstraction to analyze instructional tasks for the purpose of anticipating student reasoning and learning. We note, however, that this viability is fragile in one particular way.

A mathematics classroom is a complex, dynamic system that makes developing characterizations of cognitive activity a complicated pursuit. The complexity of a classroom yields significant limitations as it relates to gaining insights into and developing evidence for particular abstractions. In our analysis, we were constrained to drawing inferences from the observable behaviors and utterances in the classroom, and these are imperfect in their representation of cognitive activity. As a case-in-point, our analysis of Ms. R's implementation suggests the classroom interactions foregrounded pseudo-empirical abstractions involving associations between numerical patterns and features of formulas. However, it could have been the case that the students held in mind particular structured patterns of growth and the enactment of those structures in the associated geometric contexts. Unfortunately, the nature of the interactions between the students and teacher did not afford us such insights. Our inferences with respect to the classroom interactions were largely informed by our expertize and extant research, and it is possible that numerous alternative abstractions were in process or occurred during the class.

Based on the classroom implementations we analyzed, we interpret the aforementioned limitation to not solely be a methodological one. The limitation also underscores the difference between pseudo-empirical and reflecting abstractions as it relates to supporting the latter in the classroom. Pseudo-empirical abstractions can form critical springboards to student's mathematical development and they can be useful meanings themselves (Ellis, Lockwood, & Ozaltun-Celik, 2022; Ellis et al., 2024; Steffe & Olive, 2010), yet they can be less productive or generative than reflecting abstractions. This is because the focus of pseudo-empirical abstractions is primarily on the results of activity and potentially tied to carrying out the activity itself, while reflecting abstractions necessitate coordinating the results of activity with the operations driving that activity. For this reason, pseudo-empirical abstractions can become a natural focus in the classroom, whether intended or not, due to the natural inclination to focus on results of activity; results are often the more salient aspect of activity. Supporting reflecting abstractions requires that students not only engage in actions that warrant reflecting abstractions, but that their classroom experiences push them past a focus on results so that they take their actions as objects of reasoning (Simon, 2014; Simon et al., 2010). This is far from a trivial phenomenon, and thus unlikely to occur haphazardly across a majority of students in a classroom. It is also difficult (and often unfeasible) to give attention to *each* student's reasoning in a way that a teacher can have confidence in their assessment of their students' abstractions.

Before closing, we underscore that the reader should not interpret our analysis as a criticism of the teachers who participated in this study, nor is our analysis a criticism of their instruction or tasks. Teaching is a complicated activity mitigated by a number of factors extending beyond that of targeted abstractions. Returning to Ms. R, her curriculum and implementation is typical not only of algebra instruction, but also most curricula do not encourage the construction of a constantly-changing rate of change and coordinating that with geometric properties of growth. We have no evidence of Ms. R being intentionally negligent with respect to abstraction, and our use of abstraction and subsequent analysis is best viewed as a researcher's guide for developing characterizations for learning in a classroom.

6. Future work

Our results indicate the difficulty of supporting reflecting abstraction, particularly in a classroom with a typical number of students. Based on our educational experiences and careers, we do not believe such difficulties are unique to the classrooms we observed. Our results indicate the need for future work that yields insights into the ways in which to support reflecting (and reflected) abstractions in the classroom. In alternative work we have engaged in a related pursuit with respect to classroom supports for generalization (Ellis et al., 2024), and we envision a compatible line of future work that directly targets abstraction and its viability for investigating task design and implementation in a few ways.

Firstly, the current report is limited to two content areas. Future work should look to extend the framing to other content areas. Secondly, our analysis of the instructional tasks consists of *hypotheses* and, as mentioned in the discussion section, the classroom implementation data made it difficult to gain insights into students' realized abstractions. A more holistic account should include a focus on students' realized abstractions, as well as the potential role of teacher knowledge, teacher beliefs, and instructional moves in students' construction of those abstractions. For instance, we envision researchers pairing classroom data collection with clinical interviews (Ginsburg, 1997) for the purpose of developing more fine-grained characterizations of students' realized abstractions. These realized abstractions can then be compared to those inferred from classroom data. Furthermore, we envision researchers exploring alternative methodologies in the classroom that might be more amenable to analyzing student abstraction. Whereas we intended to have minimal influence on the classroom environment, other methods exist that involve researchers or coaches working side-by-side with teachers (e.g., Munson & Dyer, 2023). These methods can vary in their balance between probing and intervention, and we envision they offer an opportunity to gain deeper insights into students' realized abstractions via working with a teacher to elicit evidence for those abstractions.

Thirdly, we envision that the forms of abstraction can provide a cognitive-focused approach to modifying instructional tasks and their implementation. On the surface, pseudo-empirical and reflecting abstraction provide binary constructs in that they offer two forms by which to differentiate student reasoning, whether anticipated or realized. Reflecting abstraction and pseudo-empirical abstraction need not be positioned as binary constructs acting as alternatives to each other. As mentioned above, pseudo-empirical abstractions can provide the source material for subsequent reflecting abstractions, and reflecting abstractions can be supportive of pseudo-empirical abstractions during subsequent learning (Ellis et al., 2024; Piaget, 2001). Thus, a critical next step to the work presented here is to consider both task and implementation analyses in a way that is sensitive to how learning might unfold over time through iterative abstraction processes. For instance, based on analysis like that provided here and then subsequent investigations into students' realized abstractions and aspects of instruction contributing to those abstractions, researchers and teachers can look to modify instructional tasks to better reflect students' realized abstractions and associated processes. In the presence of a particular pseudo-empirical abstraction, future work might identify ways to generate the perturbations and accommodations necessary for the realization of reflecting abstractions. We envision that more exhaustive descriptions of tasks in terms of students' iterative abstraction processes can also provide a guide for investigating fidelity as it relates to the implementation of those tasks and supporting productive abstraction in the classroom.

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References

Antonides, J., & Battista, M. T. (2022). A learning trajectory for enumerating permutations: Applying and elaborating a theory of levels of abstraction. *The Journal of Mathematical Behavior, 68*, Article 101010. https://doi.org/10.1016/j.jmathb.2022.101010

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Information Age.

Bruner, J.S. (1966). Toward a theory of instruction Belkapp Press.

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *The Journal of Mathematical Behavior, 48*, 168–193.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378. https://doi.org/10.2307/4149958

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3–20. ⟨http://www.jstor.org/stable/1001789⟩.

Doyle, W. (1983). Academic work. *Review of Educational Research, 53*(2), 159–199. https://doi.org/10.3102/00346543053002159

Doyle, W. (1988). Work in mathematics classes: The Context of Students' Thinking During Instruction. *Educational Psychologist, 23*(2), 167–180. https://doi.org/10.1207/s15326985ep2302_6

Dreyfus, T. (2014). Abstraction in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 5–8). Springer Netherlands. https://doi.org/10.1007/978-94-007-4978-8_2.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D.O. Tall (Ed.), Advanced mathematical thinking (pp. 95–126). Kluwer. https://doi.org/10.1007/0–306-47203–1_7.

Ellis, A. B. (2007a). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education, 38*(3), 194–229. ⟨http://www.jstor.org/stable/30034866⟩.

Ellis, A. B. (2007b). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439–478.

Ellis, A. B. (2007c). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. *Journal of the Learning Sciences, 16*(2), 221–262. https://doi.org/10.1080/10508400701193705

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education, 42*(4), 308–345. ⟨http://www.eric.ed.gov/ERICWebPortal/detail?accno=EJ932248⟩.

Ellis, A.B., (2022). Decentering to build asset-based learning trajectories. In A.E. Lischka, E.B. Dyer, R.S. Jones, J.N. Lovett, J. Strayer & S. Drown (Eds.), Proceedings of the forty-fouth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 15–29). Middle Tennessee State University.

Ellis, A. B., Ely, R., Singleton, B., & Tasova, H. (2020). Scaling-continuous variation: Supporting students' algebraic reasoning. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-020-09951-6

Ellis, A. B., & Grinstead, P. (2008). Hidden lessons: How a focus on slope-like properties of quadratic functions encouraged unexpected generalizations. *The Journal of Mathematical Behavior, 27*(4), 277–296. https://doi.org/10.1016/j.jmathb.2008.11.002

Ellis, A. B., Lockwood, E., & Ozaltun-Celik, A. (2022). Empirical re-conceptualization: From empirical generalizations to insight and understanding. *The Journal of Mathematical Behavior, 65*, Article 100928. https://doi.org/10.1016/j.jmathb.2021.100928

Ellis, A. B., Lockwood, E., Tillema, E., & Moore, K. C. (2022). Generalization across multiple mathematical domains: Relating, forming, and extending. *Cognition and Instruction, 40*(3), 351–384. https://doi.org/10.1080/07370008.2021.2000989

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior, 39*, 135–155. https://doi.org/10.1016/j.jmathb.2015.06.004

Ellis, A. B., Paoletti, T., & Lockwood, E. (2024). Empirical and reflective abstraction. In P. C. Dawkins, A. J. Hackenberg, & A. Norton (Eds.), *Piaget's genetic epistemology for mathematics education research. Research in mathematics education* (pp. 169–208). Cham: Spring. https://doi.org/10.1007/978-3-031-47386-9_6.

Ellis, A. B., Tillema, E., Lockwood, E., & Moore, K. C. (2017). Generalization across domains: The relating-forming-extending framework. In E. Galindo, & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American chapter of the international group for the psychology of mathematics education* (pp. 677–684). Hoosier Association of Mathematics Teacher Educators.

Ellis, A. B., Waswa, A., Tasova, H. I., Hamilton, M., Moore, K. C., & Çelik, A. (2024). Classroom supports for generalizing. *Journal for Research in Mathematics Education, 55*(1), 7–30. https://doi.org/10.5951/jresematheduc-2022-0140

Flores, M. M. (2009). Using the concrete-representational-abstract sequence to teach subtraction with regrouping to students at risk for failure. *Remedial and Special Education, 31*(3), 195–207. https://doi.org/10.1177/0741932508327467

Fonger, N. L., Ellis, A. B., & Dogan, M. F. (2020). A quadratic growth learning trajectory. *The Journal of Mathematical Behavior, 59*, Article 100795. https://doi.org/10.1016/j.jmathb.2020.100795

Gehrke, N. J., Knapp, M. S., & Sirotnik, K. A. (1992). In search of the school curriculum. *Review of Research in Education, 18*(1), 51–110. https://doi.org/10.3102/0091732X018001051

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press,.

von Glasersfeld, E. (1991). Abstraction, re-presentation, and reflection: An interpretation of experience and of Piaget's approach. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 45–67). Springer. https://doi.org/10.1007/978-1-4612-3178-3_4.

von Glasersfeld, E. (1995). *Radical constructivism: A Way of knowing and Learning Falmer Press*. https://doi.org/10.4324/9780203454220

Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 383–432.

Harel, G.,(2001). The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Instruction.

Hinton, V. M., & Flores, M. M. (2019). The effects of the concrete-representational-abstract sequence for students at risk for mathematics failure. *Journal of Behavioral Education, 28*(4), 493–516. https://doi.org/10.1007/s10864-018-09316-3

Howson, A. G., Keitel, C., & Kilpatrick, J. (1981). *Curriculum development in mathematics*. Cambridge University Press,.

Kilpatrick, J. (2011). Slouching toward a national curriculum. *Journal of Mathematics Education at Teachers College, 2*, 1.

Knuth, E. (2000). Student understand of the Cartesian Connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500–508.

Liang, B.,(2021). Learning about and learning from students: Two teachers' constructions of students' mathematical meanings through student-teacher interactions [Ph.D. Dissertation].

Liang, B., & Moore, K. C. (2021). Figurative and operative partitioning activity: A student's meanings for amounts of change in covarying quantities. *Mathematical Thinking & Learning, 23*(4), 291–317.

Lobato, J., Ellis, A. B., & Munoz, R. (2003). How "focusing phenomena" in the instructional environment support individual students' generalizations. *Mathematical Thinking & Learning, 5*(1), 1–36.

Miller, S. P., Stringfellow, J. L., Kaffar, B. J., Ferreira, D., & Mancl, D. B. (2011). Developing computation competence among students who struggle with mathematics. *Teaching Exceptional Children, 44*(2), 38–46. https://doi.org/10.1177/004005991104400204

Moore, K. C. (2014). Signals, symbols, and representational activity. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 211–235). University of Wyoming.

Moore, K. C. (2021). Graphical shape thinking and transfer. In C. Hohensee, & J. Lobato (Eds.), *Transfer of learning: Progressive perspectives for mathematics education and related fields* (pp. 145–171). Springer.

Moore, K.C., Liang, B., Stevens, I.E., Tasova, H.I., & Paoletti, T. (in press). Abstracted quantitative structures: Using quantitative reasoning to define concept construction In G.K. Akar, O.Z. Ismail, S. Arslan, & P.W. Thompson (Eds.), Quantitative reasoning in mathematics and science education. Springer.

Moore, K. C., Liang, B., Stevens, I. E., Tasova, H. I., & Paoletti, T. (2022). Abstracted quantitative structures: Using quantitative reasoning to define concept construction. In G. Karagöz Akar, İ.Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative reasoning in mathematics and science education* (pp. 35–69). Springer International Publishing. https://doi.org/10.1007/978-3-031-14553-7_3.

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior, 56*. https://doi.org/10.1016/j.jmathb.2019.01.008

Munson, J., & Dyer, E. B. (2023). Pedagogical sensemaking during side-by-side coaching: Examining the in-the-moment discursive reasoning of a teacher and coach. *Journal of the Learning Sciences, 32*(2), 171–210. https://doi.org/10.1080/10508406.2022.2132863

Norton, A., & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fraction schemes. *The Journal of Mathematical Behavior, 28*(2–3), 150–161. https://doi.org/10.1016/j.jmathb.2009.06.002

Oehrtman, M. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In M. P. Carlson, & C. L. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 65–80). Mathematical Association of America.

Paoletti, T. (2020). Reasoning about relationships between quantities to reorganize inverse function meanings: The case of Arya. *The Journal of Mathematical Behavior, 57*, Article 100741. https://doi.org/10.1016/j.jmathb.2019.100741

Peterson, S. K., Mercer, C. D., & O'Shea, L. (1988). Teaching learning disabled students place value using the concrete to abstract sequence. *Learning Disabilities Research, 4*, 52–56.

Piaget, J. (1970). *Genetic epistemology*. W. W. Norton & Company, Inc,.

Piaget, J. (1977). *Recherches sur l'abstraction réfléchissante* (Vol. I & II). Presses univ. de France.

Piaget, J. (2001). *Studies in reflecting abstraction*. Psychology Press Ltd,.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211–246. ⟨http://www.eric.ed.gov/ERICWebPortal/detail?accno=EJ737280⟩.

Remillard, J.T., Herbel-Eisenmann, B., & Lloyd, G.M.,(2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials.

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th annual meeting of the North American chapter of the international group for the psychology of mathematics education* (Vol. 1, pp. 298–303). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Schoenfeld, A.H.,(1991). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J.F. Voss, D.N. Perkins, & J.W. Segal (Eds.), Informal Reasoning and Education (pp. 311–343).

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification-The case of function. In G. Harel, & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA Notes* (Vol. 25, pp. 59–84). Mathematical Association of America.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*, 499–511.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145. ⟨http://www.eric.ed.gov/ERICWebPortal/detail?accno=EJ504021⟩.

Simon, M. A. (2014). An emerging theory for design of mathematical task sequences: Promoting reflective abstraction of mathematical concepts. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education* (Vol. 5, pp. 193–199). PME.

Simon, M. A., Placa, N., & Avitzur, A. (2016). Participatory and anticipatory stages of mathematical concept learning: Further empirical and theoretical development. *Journal for Research in Mathematics Education, 47*(1), 63–93. https://doi.org/10.5951/jresematheduc.47.1.0063

Simon, M. A., Saldanha, L. A., McClintock, E., Akar, G. K., Watanabe, T., & Zembat, I. O. (2010). A developing approach to studying students' learning through their mathematical activity. *Cognition and Instruction, 28*(1), 70–112. https://doi.org/10.1080/07370000903430566

Steffe, L. P., & D'Ambrosio, B. S. (1995). Toward a working model of constructivist teaching: A reaction to simon. *Journal for Research in Mathematics Education, 26*(2), 146–159. ⟨http://www.eric.ed.gov/ERICWebPortal/detail?accno=EJ504022⟩.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. Springer,.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh, & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Erlbaum.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455–488. https://doi.org/10.2307/1163292

Stein, M. K., Remillard, J. T., & Smith, M. (2007). How curriculum influences student learning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319–369). Information Age.

Stylianides, A. J., & Stylianides, G. J. (2008). Studying the classroom implementation of tasks: High-level mathematical tasks embedded in 'real-life' contexts. *Teaching and Teacher Education, 24*(4), 859–875. https://doi.org/10.1016/j.tate.2007.11.015

Tallman, M.A. (2015). An examination of the effect of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge [Unpublished Ph.D. dissertation].

Tallman, M. A., & Frank, K. M. (2020). Angle measure, quantitative reasoning, and instructional coherence: an examination of the role of mathematical ways of thinking as a component of teachers' knowledge base. *Journal of Mathematics Teacher Education, 23*, 69–95.

Tallman, M. A., & O'Bryan, A. E. (2024). Reflected abstraction. In P. C. Dawkins, A. J. Hackenberg, & A. Norton (Eds.), *Piaget's genetic epistemology for mathematics education research. Research in mathematics education* (pp. 239–288). Cham: Springer. https://doi.org/10.1007/978-3-031-47386-9_8.

Tasova, H. (2021). Developing middle school students' meanings for constructing graphs through reasoning quantitatively [Ph.D. Dissertation]. University of Georgia: USA.

Teuscher, D., Moore, K. C., & Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education, 19*(5), 433–456. https://doi.org/10.1007/s10857-015-9304-0

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165–208.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel, & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*. SUNY Press.

Thompson, P. W. (1994b). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2-3), 229–274. https://doi.org/10.1007/BF01273664

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–49). PME.

Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education (pp* (pp. 57–93). Springer. https://doi.org/10.1007/978-1-4614-6977-3_4.

Tillema, E. S. (2014). Students' power meanings of multiplication. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 281–301). University of Wyoming.

Tillema, E. S., & Hackenberg, A. J. (2011). Developing systems of notation as a trace of reasoning. *For the Learning of Mathematics, 31*(3), 29–35.

Tzur, R., & Simon, M. A. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education, 2*(2), 287–304.

Waswa, A. (2023). Investigating secondary preservice teachers' mathematical creativity: The context of quadratic growth [Ph.D. Dissertation]. University of Georgia: USA.

Wittrock, M. C. (1987). Teaching and Student Thinking. *Journal of Teacher Education, 38*(6), 30–33. https://doi.org/10.1177/002248718703800606

Witzel, B. S., Ferguson, C. J., & Mink, D. V. (2012). Number sense: Strategies for helping preschool through grade 3 children develop math skills. *Young Children, 67*, 89–94.