Covariational Reasoning and Invariance Among Coordinate Systems

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A B S T R A C T

Researchers continue to emphasize the importance of covariational reasoning in the context of students' function concept, particularly when graphing in the Cartesian coordinate system (CCS). In this article, we extend the body of literature on function by characterizing two pre-service teachers' thinking during a teaching experiment focused on graphing in the polar coordinate system (PCS). We illustrate how the participants engaged in covariational reasoning to make sense of graphing in the PCS and make connections with graphing in the CCS. By foregrounding covariational relationships, the students came to understand graphs in different coordinate systems as representative of the same relationship despite differences in the perceptual shapes of these graphs. In synthesizing the students' activity, we provide remarks on instructional approaches to graphing and how the PCS forms a potential context for promoting covariational reasoning.

1. Introduction

First introduced at the elementary level, graphs are essential representations for the study of numerous mathematical topics including modeling relationships between quantities, exploring characteristics of functions, solving for unknown values, and investigating geometric transformations. Highlighting the central role of graphing in mathematics education, the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices Council of Chief State School Officers, 2010) contains some form of the term *graph* on more than a third of the document's pages. Building from the emphasis on graphing at the K-12 level, graphing is central to the study of several undergraduate mathematics courses: calculus, differential equations, and analysis, to name a few.

Reflecting the heavy focus on graphing in school mathematics, mathematics education research has given significant attention to graphing, with a multitude of studies (e.g., Carlson, 1998; Leinhardt, Zaslavsky, & Stein, 1990; Oehrtman, Carlson, & Thompson, 2008) having investigated students' meanings for graphing in the context of function. Although graphing receives significant attention in mathematics education research, little of this focus has been given to graphing in the *polar coordinate system* (PCS). Complicating the matter, the sparse research (Montiel, Vidakovic, & Kabael, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009; Sayre & Wittman, 2007) available on students' meanings for the PCS highlights student difficulties, particularly pointing to difficulties involving problematic connections with the *Cartesian coordinate system* (CCS).

In the present study, we explore students' thinking when graphing in the PCS and draw connections with existing research on graphing and function. Specifically, we discuss two undergraduate students' reasoning when graphing in the PCS. To graph relationships in the PCS, both students engaged in several ways of thinking that ranged from determining and plotting

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discrete points to reasoning about how quantities continuously vary in tandem. We illustrate these ways of thinking across several different tasks and draw attention to the implications of these ways of thinking relative to the students' ability to flexibly use the PCS and CCS to represent relationships between quantities. Against the backdrop of these findings, we conclude by discussing how the PCS offers a *potential* setting for promoting quantitative and covariational reasoning.

2. Background

Often first introduced in a precalculus course, the PCS is critical for the study of advanced mathematics and can be found in numerous applications in engineering and the sciences. The PCS also plays a central role in exploring complex numbers. For instance, the operations of multiplication, division, and exponentiation are more readily explored when using the polar form of complex numbers. Although the PCS plays an important role in the aforementioned areas, research on student thinking in the context of the PCS is sparse, with a pair of studies by Montiel et al. (2008, 2009) forming the most applicable works to the present study. Both studies included a focus on the PCS, with the earlier study (Montiel et al., 2008) having explored relationships among two-dimensional coordinate systems and the subsequent study (Montiel et al., 2009) having included two- and three-dimensional coordinate systems. The authors made several important observations across these studies including how students' function meanings can cause difficulties when extended to the PCS.

Of relevance to the present study, Montiel et al. (2008) identified that the connections students create between the CCS and PCS are tied to their meanings for function and graphing in the CCS. For instance, students often relied on rules learned in the context of the CCS to determine whether a given relation is a function. These rules included applying the vertical line test to determine whether a graphed relationship in the PCS is a function. Some students also referenced "known" functions when determining whether graphs were functions. By "known," we interpret the authors to mean that the students recalled a shape in the plane that they had previously deemed a function (e.g., a student claiming that a parabola opening down is a function because parabolas are defined as such). Compatible with the earlier study (Montiel et al., 2008), Montiel et al. (2009) noted that when students moved among representational systems, the students' function meanings did not entail coordinating the different conventions of the representational systems, with the students often relying on the conventions from one representational system (e.g., the CCS).

Montiel and colleagues' findings (Montiel et al., 2008, 2009), and specifically students' difficulty in coordinating representational systems, speak to several researchers' (Lobato & Bowers, 2000; Thompson, 1994c, 2013) statements about multiple representations. Lobato and Bowers (2000) questioned, "...whether tables, graphs, and equations are multiple representations of *anything* to students" (p. 4). Thompson (1994c) explained:

> Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us, but instead may be, as Moschkovich et al. (1993) have said, areas of representational activity among which we have built rich and varied connections...I agree with Kaput (1993) that it may be wrongheaded to focus on graphs, expressions, or tables as representations of function, but instead focus on them as representations of something that, from the students' perspective, is *representable*, such as some aspect of a specific situation. The key issue then becomes twofold: (1) to find situations that are sufficiently propitious for engendering multitudes of representational activity and (2) orient students to draw connections among their representational activities in regard to the situation that engendered them. (Thompson, 1994c, pp. 39–40)

Moreover, if students are to conceive multiple representations of *something*, then it is necessary that they not only construct the *something* that is to be represented, but also have meanings for the representational systems such that when the students operate within and move among systems, they can think about their representational activity as conveying the same *something*. Returning to the studies by Montiel and colleagues, the students did not appear to have distinct meanings for the coordinate systems that simultaneously supported connections among the systems. Instead, their meanings were inherently tied to the conventions of one coordinate system (e.g., the vertical line test and the CCS). In short, representations are not external to the person doing the representing but instead consist of a system of mental actions and meanings that the individual has organized into some cognitive structure (von Glasersfeld, 1987).

The construct of covariational reasoning (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998) foregrounds students' construction of "the something that, from the students' perspective, is *representable*." Covariational reasoning entails the mental actions involved in conceiving two quantities as varying in tandem (Carlson et al., 2002; Saldanha & Thompson, 1998) and is central to students' understanding of numerous secondary mathematics topics including quadratic relationships (Ellis, 2011), exponential relationships (Castillo-Garsow, 2010; Confrey & Smith, 1995), trigonometric relationships (Moore, 2012), rate of change (Carlson et al., 2002; Thompson, 1994a), function (Oehrtman et al., 2008), and the Fundamental Theorem of Calculus (Thompson, 1994b).

In characterizing second-semester calculus students' thinking, Carlson et al. (2002) identified how the students' covariational reasoning influenced their ability to make sense of dynamic situations, interpret graphs, and create graphs. Specifically, the authors identified several mental actions that the students engaged in when coordinating quantities that vary in tandem. These mental actions included, but were not limited to, coordinating directional change (e.g., quantity *A* increases while quantity *B* increases), coordinating amounts of change (e.g., the increase in quantity *A* decreases for successive increases

in quantity *B*), and coordinating rates of change (quantity *A* increases with a decreasing rate with respect to an increasing quantity *B*).

The tasks found in Carlson et al.'s (2002) study involved the CCS, but the mental actions associated with covariational reasoning need not be specific to the coordinate system in which one is graphing (nor are they specific to the act of graphing). To illustrate, consider graphing the relationship defined by $f(b)=b^2$. For the function *f* and $b > 0$, as the *input* increases, the *output* increases at an increasing rate. It follows that the *output* increases such that the *change of output* increases for successive equal *changes of input*. These *change of output* values increase by a constant amount for successive equal *changes of input*; as the *input changes* from 0 to 1 to 2 to 3 and so on, the *output increases by* 1, 3, 5, and so on. Hence, the *change of output* increases by 2 for each successive *change of input* of 1. The aforementioned covariational relationship can be represented in both the CCS ($y = x^2$, Fig. 1, top) and PCS ($r = \theta^2$, Fig. 1, bottom). Although changing coordinate systems results in a different pictorial object, covariational reasoning enables conceiving the graphs as one in the same; changing the coordinate system changes the spatial conventions by which we represent two quantities and thus the shape of the curve, but the defined covariational relationship remains invariant. It is in this way that the graphs and formula can represent the same *something* – a dynamic structure of related quantities.

We note that another common graphing topic is determining both the PCS and CCS mappings that define a specified curve in a plane (e.g., Fig. 2). In such a situation, the invariant object is a curve in the plane (in a perceptual sense), where the curve defines different relationships depending on the coordinate system of choice. This differs from the prior discussion that focused on a covariational relationship as the invariant object (Fig. 1); whereas the situation presented in Fig. 1 foregrounds a covariational relationship, the situation conveyed by Fig. 2 foregrounds a curve in the plane. For example, when tasked with determining whether $r = 2$ defines a function, one student in the Montiel et al. (2008) study first determined the CCS representation of the curve defined by $r = 2$, namely $x^2 + y^2 = 4$. The student then used the CCS representation and formula to claim that $r = 2$ is not a function. While the authors presented both a curve and formula in their problem, if only presented
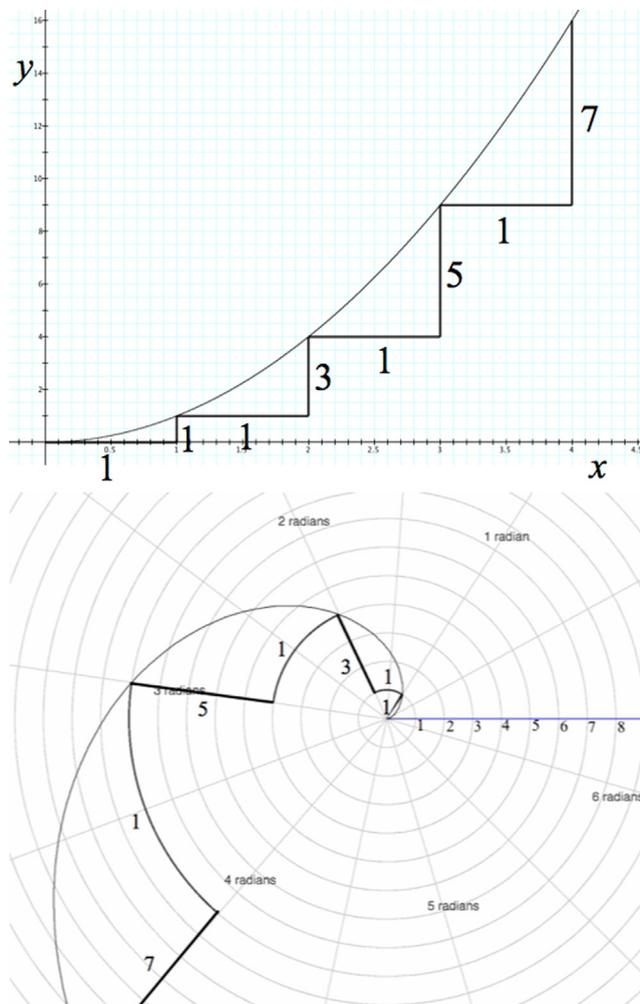

Fig. 1. Representing a covariational relationship: $y = x^2$ (top) and $r = \theta^2$ (bottom).
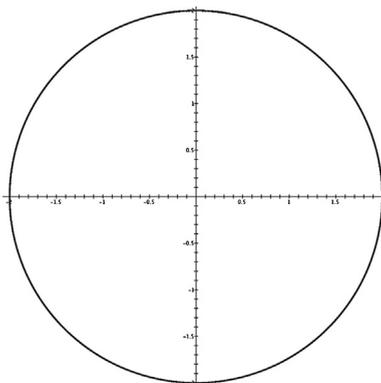
Fig. 2. A curve in the plane defined by $r = 2$, $x^2 + y^2 = 4$, and $(x(t), y(t)) = (2\cos(t), 2\sin(t))$.

with a curve in the plane (e.g., Fig. 2) and asked to determine whether the curve defines a function, we might expect that particular student to respond in a similar way (e.g., basing his answer on the CCS to claim that the curve does not define a function). Reconciling this problem requires that the student understand the ambiguous nature of questioning whether a curve in a plane defines a function – the answer depends upon the coordinate system of choice, as different coordinate systems correspond to different relationships. It is not the curve that is or is not a function, but instead the relationship defined by the curve, where this relationship is dependent on the chosen coordinate system.

Of relevance to the present study, several researchers (Castillo-Garsow, 2010, 2012; Castillo-Garsow, Johnson, & Moore, submitted for publication) have recently characterized two different images of change associated with covariational reasoning: *chunky* and *smooth*. As Castillo-Garsow (2010, 2012) and Castillo-Garsow et al. (submitted for publication) described, chunky images of change entail a nearly singular focus on discrete points, which the authors likened to imagining how the volume and height of water in a bottle varies when filling the bottle cup by cup. On the other hand, smooth images of change entail considering a continuous change in progress, which the authors likened to imagining how the volume and height of water varies when filling up a bottle of water by turning on a water spigot. The emphasis of these two images is on how an individual conceptualizes variation. For instance, returning to Fig. 1, a student might conceptualize the graph in terms of one quantity increasing with an increasing intensity for an ongoing (e.g., smooth) increase in the other quantity. Another student might imagine change occurring in successive amounts (e.g., chunky), supporting their identification that for successive changes in one quantity, the change in the other quantity increases by a fixed amount. Also, a student might organize these two images of change into a coordinated structure of covarying quantities (Castillo-Garsow, 2012; Saldanha & Thompson, 1998; Thompson, 2008, 2011).

3. Methodology

In the present study, we conducted a teaching experiment with the intention of characterizing models of students' mathematics (Steffe & Thompson, 2000). We specifically sought to characterize the students' mental actions driving their representational activity. The following research questions guided the study:

1. What ways of thinking do students engage in when graphing in the PCS?
2. How do students use the CCS when graphing in the PCS?

3.1. Subjects and setting

The subjects of this study, 'Jack' and 'Kate,' were two undergraduate students enrolled in a pre-service secondary mathematics education program at a large public university in the southeast United States. At the time of data collection, the participants were third year students (by credits taken) and taking the first pair of courses (one methods and one content) in the program. We chose the students on a voluntary basis from the content course, in which the lead author was the instructor.

We chose to work with pre-service teachers for several reasons. First, the present study was situated within a series of investigations (e.g., Moore, 2012; Moore, LaForest, & Kim, 2012) into pre-service teachers' covariational and quantitative reasoning in the context of trigonometric functions and the PCS. The content course covered trigonometric functions and graphing in multiple coordinate systems, and thus the enrolled students were a natural fit for the study. Second, our experiences teaching the course during previous semesters suggested that students' ways of thinking about graphing became problematic when attempting to graph relationships in multiple coordinate systems. Thus, we intended to gain insights into these difficulties and the ways of thinking that help or inhibit students' ability to reconcile such difficulties.

The content course engaged the students in quantitative reasoning (Thompson, 1990) and covariational reasoning to explore topics central to secondary mathematics (e.g., trigonometry, exponential functions, linear functions, rate of change, and accumulation). Prior to the present study, the course explored ideas of angle measure and trigonometric functions. The approach to these topics was grounded in previous research (Moore, 2012, 2013) on students' learning of angle measure and trigonometric functions. Due to the focus on covariational reasoning in the context of trigonometric functions, we expected the students to be familiar with covariational reasoning when entering the study. Based on previous research highlighting the difficulties students have with the PCS (Montiel et al., 2008, 2009) and covariational reasoning (Carlson et al., 2002), we questioned whether they would or would not spontaneously engage in said reasoning when graphing in the PCS.

3.2. Data collection and analysis

The teaching experiment (Steffe & Thompson, 2000) consisted of five 75-min teaching sessions with the pair of students. The teaching sessions occurred during the aforementioned content course meetings. A typical teaching session included paired group work with whole class discussions following paired group work. The paired group work spanned a majority of each 75-min session. In an attempt to foster interactions in the paired setting and capture artifacts of student thinking, the pair shared a 3-foot by 3-foot whiteboard while a researcher facilitated the discussion. The whole class discussions, which included all members of the content course, provided the students an opportunity to present and consider each other's work after working in pairs, with the instructor facilitating these discussions. During the whole class discussions, the instructor and students often used graphing programs that were projected on a screen. The first teaching session developed conventions of the PCS (e.g., coordinate pairs representing the distance from a fixed point and the radian measure of an arc) and supported students' spatial reasoning in the PCS (e.g., considering the location of a point that has a varying arc measure and a constant distance measure, and vice versa). The subsequent teaching sessions, which are the focus of the present report, involved graphing functions of the form $r = f(\theta)$ or $\theta = g(r)$ in the PCS.

Data collection involved videotaping and digitizing Jack and Kate's activity during the teaching sessions. Also, fellow researchers observed each teaching session, taking notes of the interactions between the researcher and students. We debriefed immediately after each session in order to discuss the students' thinking and document all instructional decisions. Our retrospective analysis of the data involved first transcribing the entire data set and then identifying instances offering insights into the students' thinking. We then performed a conceptual analysis (Thompson, 2000) of these instances in order to generate and test models of the students' thinking so that these models provided viable explanations of their behaviors. We particularly sought to characterize the students' reasoning when graphing in the PCS and CCS.

4. Results

As described in the prior section, we first worked with the students to support their construction of the PCS.[1] Nearly two full sessions were dedicated to exploring the conventions of the PCS, including: (i) the quantitative basis for the coordinate system (e.g., a directed radial distance from a fixed point and a directed angle measure from a fixed ray); (ii) each point in the plane can be represented by an infinite number of coordinate pairs (e.g., $(r, \theta \pm 2n\pi)$ defines the same point in the plane for all integers n); and (iii) how variations in one coordinate value influence the location of a point in the plane. Of relevance to the present study, the students' actions did not suggest familiarity with the PCS. The students verbalized that they had "seen" the PCS in previous courses, but that they had not explored the aforementioned features of the PCS and that the PCS had previously confused them. We took the students' actions and comments to suggest that they had not experienced a setting that emphasized the *construction* of the PCS prior to the first two sessions.

Following the introduction to the PCS, we transitioned into graphing relationships including linear, quadratic, and trigono-metric relationships. As the students completed the proposed tasks, their solutions indicated engagement in several ways of thinking. In this section, we discuss the students' actions as they graphed relationships and interpreted graphs in the PCS.

4.1. Graphing linear relationships

We first tasked the students with graphing $f(\theta) = 2\theta + 1$, with an interest in how the students would reconcile a linear relationship not being represented by a line in the PCS. Jack and Kate initially graphed the relationship in the CCS (e.g., $y = 2x + 1$) by identifying both the x- and y-intercepts and connecting these points with a line. Determining intercepts and connecting these intercepts with a line did not carry over to the PCS, and to gain a better sense of the function they plotted points for θ values of 0, 1, 2, 3, and 4. They then connected these points (Fig. 3) and related the two graphs (Table 1).

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[1] This work is the focus of a manuscript under preparation, with the main emphasis being on problematic situations that arose when constructing the PCS. As one such example, the students' ability to coordinate angle measures as conveying an equivalence class of arcs that involves measuring in radii emerged as critical to their constructing the PCS (e.g., understanding angle measures as an equivalence class of arcs enabled students to conceive the set of arc lengths between two fixed rays as all corresponding to the same angle measure).
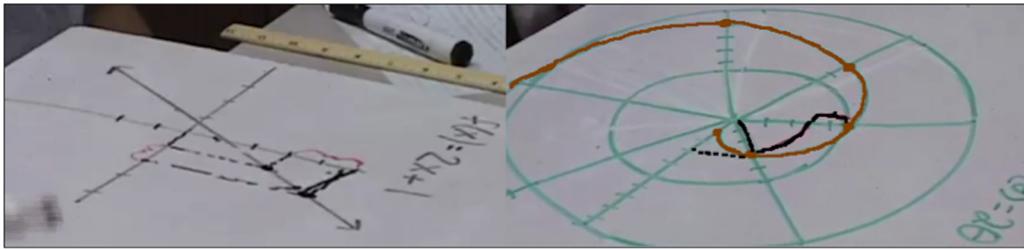
Fig. 3. Students' graphs of a "linear" relationship.

Table 1
Jack and Kate discuss two "linear" graphs.

Kate:	They went out by two, like you know here (*pointing at the two in the formula* $r = 2\theta + 1$) the slope is like two (*tapping along the CCS graph*).
Int.:	This has no slope (*pointing to the PCS graph*)...
Kate:	No, I'm relating the slope here (*pointing to the CCS graph*), to the difference in the radius of two each time (*tapping along the PCS graph*). Like [the radius is] one, three, five, seven, nine, eleven (*pointing to the corresponding points on the polar graph*), [the radius] increases by two.

Suggestive of chunky thinking, Kate reasoned that both graphs convey some quantity changing by amounts of two for successive changes of one in the other quantity. In the case of the CCS, she related this feature to "slope," meaning the amount by which *y* increased for successive equal changes in *x*. In the case of the PCS, she illustrated the increase in the distance from the pole, which she referred to as the "radius," for successive changes of angle measure.

As the interaction continued, Jack and Kate denoted the aforementioned amounts of change (Fig. 3) and claimed that both graphs convey a "constant rate of change" between the input and output values. Kate then added, "That's cool...because you'd never see this (*referring to the PCS graph*) and be like, that's a linear function." Kate's comment underscores an important aspect of the pair's thinking. From a shape standpoint, their PCS graph is not "linear." Yet, by considering the graphs in terms of covarying quantities, they conceived both graphs as representative of the same relationship (e.g., a relationship such that there is a constant rate of change between the quantities).

4.2. Graphing a quadratic relationship

Based on Jack and Kate's approach to graphing a linear relationship, we conjectured that similar reasoning would enable them to graph a relationship with a non-constant rate of change. We asked the students to graph $r = \theta^2$, which they compared to $r = \theta$. As with their solution to graphing the linear relationship, the students first plotted points and connected the points in both the CCS and PCS. They then compared their graphs (see Fig. 4 for their written products) (Table 2).

Kate first compared the perceptual shapes of the graphs, observing that the quadratic is "expanded." Such thinking is similar to a student observing one graph to be "steeper" or growing "faster" than another graph in the CCS. As the interaction continued, the students' actions suggested that their thinking was not constrained to thinking about the curves only relative to one another. Following Kate's comment, the students worked together to describe the behavior of the graphs in terms of amounts of change and rates of change between the two quantities. Specifically, the students reasoned that the graph of $r = \theta^2$ is "looser" or moves away from the pole "faster" because *r* increases at an increasing rate with respect to θ, which they confirmed by identifying specific changes in the quantities' values.

Immediately following Jack's last statement in Table 2, the students denoted amounts of change on a graph in the CCS (Fig. 4), while Kate claimed, "Like our change in input here (*referring to CCS graph*) would represent the change in this angle measure (*indicating the corresponding change on the PCS graph*), and then our output, change of radius length, and that's increasing for equal changes." Compatible with Jack and Kate's actions when graphing the linear relationship, the students
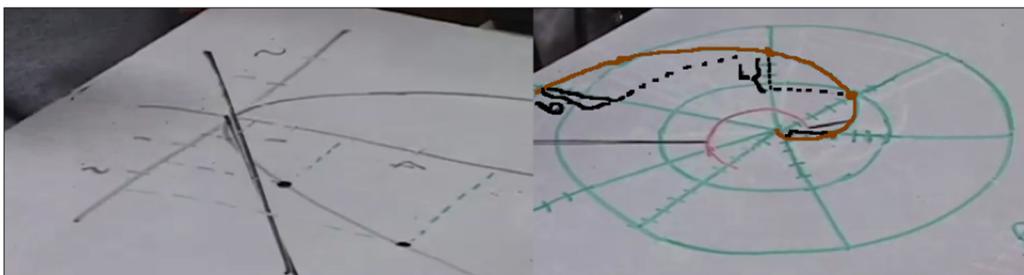

Fig. 4. Students' graphs of the quadratic relationship.

Table 2
Jack and Kate discuss a quadratic relationship.

Kate:	r of θ, but compared to r of θ^2, [r of θ^2 is] like expanded (*Kate points to the two PCS graphs and then spreads her hands apart*). Like, like, this one's (*referring to linear*) like much more tighter swirled (*moving her hands in a circular motion*) but then this one (*referring to quadratic*) is just like looser I guess.
Jack:	Yeah, we can see better, with both of them, both graphs, that the change in radius (*referring to quadratic*) for every radian further that the angle is increasing (*rotating his hand in successive rotations while spreading his index and middle finger apart*). . .Um, the radius, every time is increasing at an increasing rate (*referring to quadratic*).
Int.:	Okay now what's that mean in terms of amounts of change?
Jack:	We could do equal changes in θ and then. . ..
Kate:	Like, if we looked at first these two then these two points (*indicating the points (9, 3) to (16, 4), and then (16, 4) to (25, 5)*), the change of θ here would be this, that length (*drawing an arc from (9, 3) to (9, 4)*). But then the change is radius would be up that line (*drawing a segment from the point (9, 4) to (16, 4)*).
Jack:	Which is seven.
Kate:	And then we have the same thing (*draws an arc from (16, 4) to (16, 5) and a segment from (16, 5) to (25, 5)*). . .so you can see these black lines, the [change in radius] is increasing.
Jack:	So that's like nine to sixteen (*pointing to the segment connecting the points (9, 4) and (16, 4)*), which is seven, and this one is sixteen to twenty-five (*pointing to the segment connecting the points (16, 5) and (25, 5)*), which is nine, which we can see there too (*pointing to the CCS graph*).

conceived both graphs in terms of a structure of covarying quantities, which enabled them to understand a graph in the PCS and a graph in the CCS as one in the same; to Jack and Kate, both graphs represented the same covariational relationship.

We also note that Jack and Kate's actions on this task are suggestive of both smooth and chunky images of change. Initially, Kate's focus on the qualitative features of the graph might have entailed imagining how the radial length is increasing for an ongoing increase in angle measure. Jack's move to consider successive changes of angle measure and subsequently discuss these intervals indicates chunky thinking, while his motions (e.g., rotating his fingers while spreading apart his fingers successively) also imply that he might have imagined smooth variation over these intervals. However, at this time in the study, we were not attune to the role of chunky and smooth thinking in their activity and did not question the students further.

4.3. Covariational reasoning and conceiving a trigonometric relationship

During the first two tasks, Jack and Kate revealed a tendency to engage in covariational reasoning when graphing in both coordinate systems. Such reasoning not only enabled them to create graphs in the PCS, but it also enabled them to connect graphs in the PCS to their CCS counterparts. While the students engaged in covariational reasoning to create graphs, we questioned whether or not they would engage in such reasoning to interpret given graphs. As Montiel et al. (2008) showed, students have a tendency to rely on the CCS representations of curves in a plane. Likewise, students in previous semesters encountered much difficulty thinking about curves in a plane independent of the CCS. We conjectured that Jack and Kate's propensity to engage in covariational reasoning might result in an avoidance of these tendencies.

We tasked the students with determining a formula for a graph of the sine function ($r = \sin(\theta)$, Fig. 5). We chose the sine function to provide a curve that differed in shape from those they had explored during previous tasks, particularly in that the relationship is not monotonic for $\theta > 0$. As with the previous problems, Jack and Kate first focused on identifying specified coordinate pairs (e.g., chunky thinking). After identifying r values corresponding to θ values of 0, $\pi/2$, π, and $3\pi/2$, the students conjectured that $r = \sin(\theta)$ is the appropriate formula for the given graph. The students then drew a graph of the sine function in the CCS and justified their conjecture (Table 3).

When making sense of the graph, the students used a combination of identifying discrete points and reasoning about the graph conveying the distance from the pole as increasing or decreasing for a continuously increasing angle measure between the identified points. For instance, Kate reasoned that as the angle measure increases smoothly (e.g., a change in progress) from $3\pi/2$ radians to 2π radians, the distance from the pole decreases from a magnitude of one to zero, which corresponds
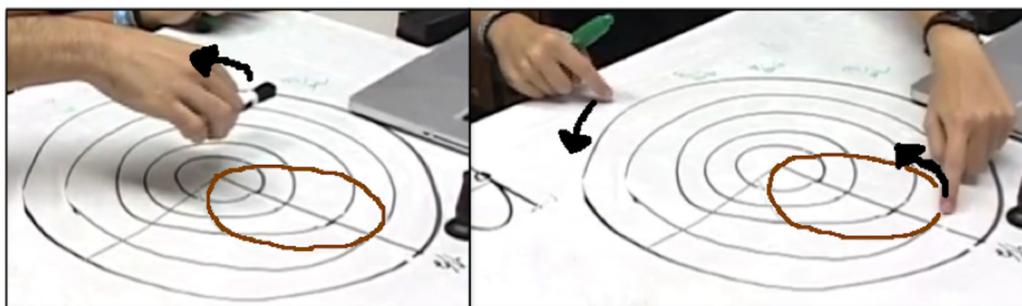

Fig. 5. Illustrating covarying quantities in a smooth way.

Table 3
Jack and Kate discuss a trigonometric relationship.

Kate:	This gets us from zero to right here is zero again (*tracing along Cartesian horizontal axis from 0 to π*). So, we start here (*pointing to the pole in the polar coordinate system*).
Jack:	Ya, and you're sweeping around because (*making circular motion with pen*), θ's increasing, distance from the origin increases and then decreases again (*Jack traces along the Cartesian graph from 0 to π as Kate traces along corresponding part of the polar graph from 0 to π*).
Int.:	OK, so you're saying as θ increases the distance from the origin does what?
Jack:	It increases until π/2 (*Kate traces along polar graph*) and then it starts decreasing (*Kate traces along polar graph as Jack traces along Cartesian graph*).
Int.:	And then what happens from like π to 2π.
Kate:	It's the same.
Jack:	Um, same idea except your, the radius is going to be negative, so it gets more in the negative direction of the angle we're sweeping out (*using marker to sweep out a ray from π to 3π/2 radians – see* Fig. 5) until 3π/2 where it's negative one away and then it gets closer to zero (*continuing to rotate marker*).
Int.:	OK, so from 3π/2 to 2π, can you show me where on this graph (*pointing to polar graph*) we would start from and end at?
Kate:	This is the biggest in magnitude, so it's the furthest away (*placing a finger on a ray defining 3π/2 and a finger at (1, π/2)*), and then [the distance from the pole] gets smaller in magnitude (*simultaneously tracing one index finger along an arc from 3π/2 to 2π and the other index finger along the graph – see* Fig. 5).

to the value increasing from −1 to 0 (Fig. 5). Likewise, Jack initially imagined the angle measure smoothly increasing from 0 to π radians to make sense of how the radial length was increasing and decreasing. Stemming from instruction occurring earlier in the semester, the students understood the sine function to convey such a relationship and thus concluded that the graph represents the relationship defined by $r = \sin(\theta)$. Additionally, following this interaction the students continued justifying their formula by identifying that the graph of $y = \sin(x)$ in the CCS conveys the same covariational relationship and has identical critical points as the PCS graph. To the students, the CCS and PCS graphs of the sine function represented the same covariational relationship.

4.4. Graphing a trigonometric relationship

To conclude the teaching experiment, we tasked Jack and Kate with graphing the relationship defined by $r = \sin(2\theta)$. Contrary to the previous tasks, they first attempted to think about the graph in terms of a transformation of $r = \sin(\theta)$. Kate described, "That just makes it out to two, doesn't it?" As she gave this explanation, she made motions to indicate the CCS graph of $y = \sin(x)$ stretching to an amplitude of two and the PCS graph changing such that the maximum distance from the pole is two. Jack did not agree with Kate, arguing that the radial value would be 0 for $\theta = \pi/2$, and thus the "function reaches 0 again at π/2, so instead of having the circle go until π radians, the circle would complete at π/2." Jack then imagined transforming the circle defined by $r = \sin(\theta)$ such that the graph is a circle contained between the rays defining the angle measure of π/2 radians.

The students did not sketch a graph at this time. Instead, and compatible with their activity on previous tasks, they moved to determining coordinate pairs choosing to use successive increments of π/4 for θ. As they determined the r values, Jack expressed, "we may not actually get shapes," and then refined their increments to intervals of π/8 for θ. Jack's actions suggest that he was concerned with how the two quantities covaried within the successive π/4 intervals and believed they should consider variation within these intervals before completing their graph.

After the students determined the r values for successive values of θ that differed by π/8, they plotted the points and connected these points (Fig. 6). The students were then asked to describe the first "petal" of the graph (Table 4).
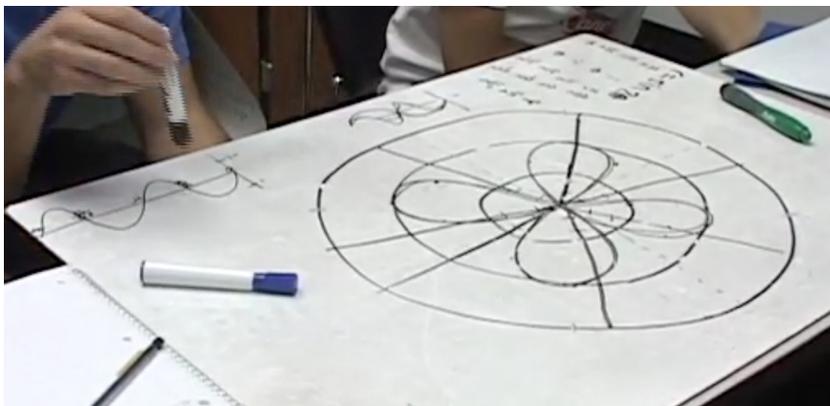

Fig. 6. Jack and Kate's board work when graphing $r = \sin(2\theta)$.

Table 4
Jack and Kate justify a petal on their graph.

Int.:	Okay good. Now let's pick this petal, so tell me why this, looks like this. What's it conveying about the relationship between r and θ for that petal?
Kate:	So the relationship is going to reverse at $\pi/4$. Right? Cause it's like. . . .
Jack:	Well, yeah cause if we look at our, at $\pi/2$ (*tracing ray*), er, $\pi/2$, on the graph of, oh, was it r equals $\sin(\theta)$. . . .
Kate:	Well like here, r is increasing (*tracing along graph from 0 to $\pi/4$ radians*) as θ increases (*tracing along arc from 0 to $\pi/4$ radians*). But then as θ is increasing (*tracing along arc from $\pi/4$ to $\pi/2$ radians*) r is decreasing (*tracing along graph from $\pi/4$ to $\pi/2$ radians*).
Jack:	Yeah, like it does at $\pi/4$ for $\sin(\theta)$ (*tracing the CCS representation of $r = \sin(\theta)$ in the air*).

Kate conceived the first petal as conveying that r increased and then decreased as θ continuously increased from 0 to $\pi/2$ radians. Kate's actions and use of the present tense suggests that she was imagining changes in progress. In trying to relate this relationship to a graph of $r = \sin(\theta)$, Jack described that both graphs have the same pattern, but over different input intervals. Following Jack's concluding statement, Kate added, "Oh, you're saying on this graph, like that (*drawing sine curve in Cartesian plane*), then r increases (*tracing increasing part of the curve*) then r decreases (*tracing decreasing part of the curve*)." Jack continued by drawing a graph of $r = \sin(2\theta)$ in the CCS to show that both $r = \sin(2\theta)$ and $r = \sin(\theta)$ convey an increasing, decreasing, increasing, decreasing, and so on pattern, to which the interviewer asked the two students to explain how the graphs of the posed relationship ($r = \sin(2\theta)$) related (Table 5).

In this case, Jack first makes the move to identify points when comparing the two graphs. Kate then described the covariation between these points in a smooth manner (e.g., using present tense to convey directional covariation) before shifting to describing particular changes in r to illustrate changing amounts of change (e.g., "like the change in r is really decreasing"). Also, throughout the interaction and compatible with the students' actions during previous interactions, Kate made smooth motions with her hands as she was describing information about the two graphs. Collectively, the students' behaviors and discussion suggests that they conceived both graphs as conveying the same relationship.

As the discussion continued past the interaction in Table 5, Jack described the covariational relationship (e.g., "r decreases at an increasing rate") for subsequent intervals of angle measure while smoothly tracing along the corresponding portions of the CCS graph. During Jack's description and tracing, Kate simultaneously traced along the corresponding part of the PCS graph, further illustrating the students' consideration of both graphs as representative of the same covariational relationship.

5. Discussion and implications

During each task, Jack and Kate engaged in several ways of thinking about relationships between two quantities' values. Both students' activity foregrounded covariational relationships, and the students conceived graphs in both the CCS and PCS as entailing these relationships. Importantly, the students' ways of thinking were not constrained to one coordinate system, nor did the students appear to give preference to one coordinate system over the other. For instance, the students did not rely on using the CCS to gain a sense of a relationship prior to graphing the relationship in the PCS. In this section, we discuss various aspects of the students' ways of thinking and then describe implications of these ways of thinking in the context of the PCS, covariational reasoning, and school mathematics.

5.1. Coordinating images of change

The various ways in which students make sense of covarying quantities are complex and have serious implications for students' learning (e.g., Carlson et al., 2002; Castillo-Garsow, 2012; Confrey & Smith, 1995; Johnson, 2012a; Thompson, 1994a,b). In Jack and Kate's case, they typically relied on first graphing discrete points and comparing discrete amounts of

Table 5
Jack and Kate relate two graphs from different systems.

Jack:	Um, well at, at $\pi/4$, the value of this graph is one (*identifying point on the CCS graph*), which is kinda how we see, when. . . .
Kate:	The radius (*horizontally moving her finger from pole to a radius of 1*) of one (*moving her finger along the arc to the point (1, $\pi/4$)*).
Jack:	This is $\pi/4$. . . .
Int.:	Okay.
Jack:	. . .the angle measure, and that's one, the radius (*identifying point on the PCS graph*). And then at $\pi/2$ the angle measure, we have zero radius here (*identifying points on both graphs*), it returns to zero (*tracing PCS graph from previous point*).
Int.:	Okay.
Kate:	And then you can tell how this increased, like, you can see, by the shape of the graph, makes sense, of the like the curve (*tracing her hand along the two curves throughout description*).
Int.:	Okay so say a little bit more about that. Why does that make sense? What is that graph showing that this graph is also showing?
Kate:	Well like here, r is increasing (*tracing CCS graph*). Like to $\pi/4$ (*pointing to $\pi/4$ on horizontal axis*), we see r is increasing here (*motioning to PCS graph*), but it's also increasing at a decreasing rate (*motioning to CCS graph*). So like here (*motioning to PCS graph*), r gets, the changes in r between these points is, like here (*identifying points at successive angle measures between 0 and $\pi/4$*), like the change in r is really decreasing. But then this is like the reverse here, it's decreasing at an increasing rate (*motioning to both graphs*).

change between these points, which is a prototypical behavior of chunky thinkers. In addition to these chunky-like behaviors, the students exhibited behaviors that are suggestive of considering a smooth change in progress within these chunks, thus illustrating that they were not constrained to thinking about discrete points and comparing discrete amounts of change. For instance, to make sense of curves between identified points the students imagined one quantity increasing or decreasing for a continuous and ongoing increase in the other quantity (e.g., Tables 3–5). Such thinking was particularly apparent when the students graphed $r = \sin(2\theta)$. During their progress on this task, Jack found it important (e.g., "we may not actually get shapes") to consider variation within already identified coordinate pairs. Jack's concern prompted the students to refine their intervals and subsequently focus on smooth variation, which led them to conceive some forms of "sameness" between relationships (e.g., patterns among $r = \sin(2\theta)$ and $r = \sin(\theta)$, and graphs $r = \sin(2\theta)$ in both the CCS and PCS).

Collectively, the students' actions highlight the importance of both chunky and smooth images of change for graphing relationships and constructing connections among coordinate systems. Even though they often focused on discrete points and amounts of change, their subsequent decisions and explanations were informed by their awareness that quantities covaried continuously in between these points; when engaging in chunky thinking, the students seemed to simultaneously hold in mind that the quantities varied smoothly within these chunks. By coordinating these images of change in different coordinate systems, the students foregrounded covariational relationships when operating under various representational conventions. Most importantly, making sense of these covariational relationships through various images of change enabled them to move among representations – both in coordinate systems and formulas – while holding in mind something invariant about these representations. It was in this way that the students were able to see different representations as one in the same. To the students, each representation represented the same *something*: a covariational relationship that entailed a structure of related quantities and various images of change.

Jack and Kate's coordination of different images of change is consistent with that of *continuous covariation* (Saldanha & Thompson, 1998; Thompson, 2008, 2011), which can be framed as a combination of smooth and chunky thinking (Castillo-Garsow, 2012; Castillo-Garsow, Johnson, & Moore, submitted for publication). An essential aspect of continuous covariation is that any conceived chunk of variation is understood as entailing smooth variation, where this smooth interval of variation can be partitioned into sub-chunks. These sub-chunks are also understood as entailing smooth variation that can be partitioned, and so on (Castillo-Garsow, 2012; Thompson, 2008, 2011). An important feature of continuous covariation, which is illustrated by Jack and Kate's actions, is that it involves more than a pointwise understanding of graphing or function. For instance, Jack and Kate not only understood that graphs in different coordinate systems involved equivalent coordinate pairs, but they also conceived invariance among the graphs in terms of a dynamic relationship between quantities' values. The students conceived graphs in terms of both discrete points and images of two dynamically changing quantities, and considering smooth variation between discrete points was central to their making sense of and producing graphs.

5.2. Polar coordinates and school mathematics

Returning to Thompson's (1994c) discussion of multiple representations, he emphasized supporting students in conceiving something that is representable, finding situations that students are likely to use multiple representations to support their thinking, and directing students to make connections among their representational activity in the context of the something that is representable. As described in the prior section, Jack and Kate's activity over the course of the study indicates that engaging in covariational reasoning supported them in conceiving something representable, namely a dynamic relationship between two quantities. Also in line with Thompson's quote, graphing in the PCS seems to have offered Jack and Kate situations that engendered multiple representations and directed them to make connections among their representational activities, with these connections being rooted in covariational reasoning.

The body of literature on covariational reasoning suggests that United States students typically do not receive repeated opportunities for covariational reasoning, nor do they experience a mathematics in which covariational reasoning is a fundamental activity. As a result, students often develop meanings for graphing and function that do not entail images of variation. For instance, Thompson (1994c), Monk (1992), and Goldenberg, Lewis, & O'Keefe (1992) documented that students are oriented to interpreting graphs in terms of pictorial objects, as opposed to interpreting graphs as relationships between two quantities' values. Thinking of the nature identified by these researchers would lead a student to conclude that the two graphs in Fig. 1 represent different relationships, as the graphs are perceptually different.

Students' propensity to treat graphs as pictorial objects is understandable if these students encountered graphing experiences that predominantly involved one coordinate system; with one coordinate system receiving focus, viewing graphs as pictorial objects solves many of the school problems that students encounter. For instance, consider an approach to quadratic functions that first presents the rule associated with the class of functions (e.g., $f(x) = ax^2 + bx + c$) and then presents a graph (or graphs) to define the prototypical shape(s) of that function class (e.g., a parabola). From here, the approach might move into plotting or estimating points, defining properties of that shape (e.g., the vertex of a parabola), and exploring these properties in terms of transformations of other members of that function class (e.g., considering the vertex in terms of translating another vertex, or how $g(x) = f(x) + 2$ can be thought of as a translation of $f(x)$). Such an approach, which is commonplace in the US, emphasizes graphs as pictorial objects made for manipulation and often spends little to no time developing graphs as representations of relationships between quantities. In the event that a student does not already understand graphs as representative of covariational relationships, it can be expected that the student will construct function meanings inherently involving the manipulation of shapes in the plane, where these manipulations must adhere to some set of rules (e.g.,

$g(x) = f(x) + 2$ means move the graph "up by 2 units"); if a student's activity is dominated by transforming shapes, then the meanings that he or she abstracts will be rooted in such activity and promote the treatment of graphs as pictorial objects made for manipulation.

Students who develop graph and function meanings rooted in shapes and operations on these shapes are posited to solve problems set within the system in which those meanings were developed. But when they move to a different system, meanings that inherently involve activity dependent on system conventions become problematic because shapes and the representational conventions are changed. For instance, if a student relies on the vertical line test as in the example provided by Montiel et al. (2008), then a circle in the plane is not a function even when defined by a rule in polar form. As another example, for a student who interprets a formula of the form $y = x + 2$ as *line* or *constant slope* (slope meaning tilt), graphing $r = \theta + 2$ in the PCS can become problematic because the graph is not a line, nor does it have a constant tilt. As Kate mentioned when graphing a constant rate of change function in the PCS, "That's cool. . .because you'd never see this (*referring to the PCS graph*) and be like, that's a linear function."

In Kate's case, conceiving the PCS graph as a linear relationship was not problematic because her reasoning included considering how two quantities covary. Likewise, both students' activity throughout the presented interactions (Tables 1–5) illustrates how covariational reasoning can give common meaning to different representational systems; as the students attempted to work within the PCS, covariational reasoning enabled seeing *something* as the same among their representational activity and was thus beneficial to the tasks at hand. Such activity illustrates that a *potential* benefit of incorporating the PCS in the study of mathematics is generating a context in which covariational reasoning can be emphasized as a productive way of thinking, particularly in the setting of using multiple representations. Investigating graphing only in the CCS has the possible consequence of reinforcing problematic student conceptions of function and graphing, many of which do not entail reasoning about covarying quantities (e.g., conceiving graphs as pictorial objects). By prompting students to consider multiple coordinate systems, a need can be established for ways of thinking (e.g., covariational reasoning) that enable conceiving graphs in each system as conveying the same relationship. As such, graphing in both systems might foster abstractions stemming from various operations involved in covariational reasoning (e.g., rate of change reasoning and coordinating amounts of change). For instance, for Kate, graphing in both coordinate systems seemed to foreground the "constant rate of change" of a linear relationship (see Table 1), as opposed to the visual tilt of a line.

While the potential use of the PCS to promote covariational reasoning is a conjecture on our part, our conjecture does have foundations in learning theory, and particularly that of abstraction (Piaget, 1970, 1997, 2001). As Oehrtman (2008) characterized when discussing abstraction and the instruction of limit concepts, abstraction theory has the following implications: (i) students' activity should reflect the structure of the intended understandings, (ii) these actions should be repeated and coordinated in a reflective way, and (iii) the development of a concept or way of thinking continually evolves through the re-construction of that concept or way of thinking in new situations. It is possible that infusing the PCS into instructional approaches to graphing can support a focus on covariational reasoning that is consistent with such a theory. Returning to the situation in which students' graphing experiences lead to meanings that are tied to activity dependent upon the representational conventions (e.g., treating curves as pictorial objects for manipulation), it is likely that they will face perturbations when moving to the PCS (e.g., a linear relationship is not represented by a line in the PCS), thus creating a need for alternative ways of thinking about graphs in both the CCS and PCS. Covariational reasoning offers one such a way of thinking, and in such a case has the potential to change one's understanding of graphing in CCS while also supporting graphing in the PCS. In the case that a student already predominantly engages in covariational reasoning when operating in the CCS, then introducing the PCS will provide a situation in which they can repeat and coordinate these actions, while reflecting on such activity in the context of graphing in the CCS. In both cases, including multiple coordinate systems provides an opportunity to engage students in multiple representational activities and support their abstraction of meanings rooted in the connections they make among these activities.

6. Looking forward

Whereas the National Council of Teachers of Mathematics's *Curriculum and Evaluation Standards for School Mathematics* (1989) and *Principles and Standards for School Mathematics* (2000) both mentioned the importance of graphing in multiple coordinate systems, as currently written the CCSSM (National Governors Association Center for Best Practices Council of Chief State School Officers, 2010) only outlines graphing relationships in the CCS. One could argue that the removal of the PCS is the appropriate course of action since students encounter well-documented difficulties with the CCS and instruction on the PCS tends to be shallow and poorly understood by students. On the other hand, not including occasions for working with other coordinate systems, and specifically the PCS, might deprive students of valuable learning opportunities. For instance, having students experience multiple coordinate systems may help them develop a deeper understanding of the CCS, possibly addressing some of the common issues seen in the literature (e.g., viewing graphs solely as pictorial objects to be manipulated). Furthermore, numerous researchers have highlighted the pedagogical benefits and importance of using multiple representations to support student learning (e.g., Goldenberg, 1995; Oehrtman et al., 2008; Romberg, Fennema, & Carpenter, 1993). Future research is needed to understand the potential use of multiple coordinate systems to support student learning, and such research should provide a finer-grained characterization of students' mental actions as they construct these coordinate systems and attempt to coordinate their activity among representational systems. The present study

illustrates that using multiple coordinate systems can aid pre-service teachers in foregrounding covariational reasoning, and subsequent studies should involve additional pre-service teachers and secondary or undergraduate mathematics students.

Our goal of the present study was to investigate student thinking when graphing in the PCS, with specific attention given to their use (or lack thereof) of covariational reasoning. Based on our previous teaching of the course and interactions with the students earlier in the semester, a covariational approach to graphing and function was not something familiar to the students. Such observations speak to the reported lack of emphasis on covariational reasoning in school mathematics (Carlson et al., 2002; Oehrtman et al., 2008). Because of this, we did not expect the students to spontaneously engage in covariational reasoning during the study. Standing in stark contrast with this conjecture, we found that the students engaged in covariational reasoning to make sense of and conceive invariant relationships among multiple representations. This leads to the question: what fostered such reasoning?

We do not argue that the tasks in the present study caused covariational reasoning per se, as the tasks themselves were not novel or designed with an explicit focus on covariational reasoning. But, graphing in the PCS and comparing this to graphing in the CCS did create a setting in which covariational reasoning enabled the students to engage in meaning making and construct connections among their activity. As several researchers have argued, students' covariational reasoning is a highly complex process that requires time and the proper support (e.g., Carlson et al., 2002; Castillo-Garsow, 2012; Confrey & Smith, 1995; Ellis, 2011; Oehrtman et al., 2008; Saldanha & Thompson, 1998). While researchers have highlighted how critical covariational reasoning is for sophisticated images of major calculus ideas like rate of change and accumulation, the same researchers have stressed that educators must not hold off on developing such reasoning until calculus (Carlson et al., 2002; Oehrtman et al., 2008; Thompson, 1994b). Thus, it becomes necessary to find places throughout school mathematics to engender covariational reasoning through rich tasks and effective instructional practices, and future studies should explore the potential of using multiple coordinate systems in that regard.

A significant limitation of this study is that it included only two students, neither of which encountered significant difficulties over the course of the teaching experiment. Because of this, the results speak to the affordances of covariational reasoning when investigating multiple coordinate systems, but not the development of such reasoning. An area of interest is thus investigating how students with a range of covariational reasoning abilities work in the PCS, and particularly those students who have not encountered instruction on quantitative and covariational reasoning. Such studies can offer deeper insights into students' *development* of covariational reasoning, including how different images of change might influence students' covariational reasoning, in the PCS. For instance, Johnson (2012a,b) has made progress in characterizing nuances in early secondary school students' covariational reasoning. Determining how such nuances evolve over time and in various contexts, including the implications of these developments, forms an important area of study. As Saldanha and Thompson (1998) described, covariational reasoning is developmental in nature (e.g., to conceive covarying quantities requires that one first conceive varying quantities to be considered in tandem), and a critical area of need for mathematics education is understanding this development and its implications for students' mathematical thinking and learning.

Acknowledgments

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