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Reasoning within quantitative frames of reference: The case of Lydia☆

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ABSTRACT

Quantitative reasoning is important in the development of K–16 mathematical ideas such as function and rate of change. Coordinate systems are used to coordinate sets of quantities by establishing frames of reference and constructing representational spaces in which sets of quantities are joined. Despite the critical role of coordinate systems in mathematics, much is left to understand about how students construct and reason within frames of reference and associated coordinate systems. In this report, we draw from a teaching experiment to discuss how an undergraduate student, Lydia, constructed and reasoned within frames of reference when graphing in non-canonical coordinate systems. We pay specific attention to distinctions between figurative and operative aspects of thought in her committing to reference points and directionality of measure comparison within frames of reference. In this regard, we present shifts in Lydia's reasoning during the teaching experiment and consider implications and future research directions.

1. Introduction

Researchers have shown that *quantitative reasoning*—reasoning about measurable attributes and how they are related (Thompson, 2011)—is important in the development of numerous K–16 mathematical ideas such as function and rate of change (Confrey & Smith, 1995; Ellis, 2007; Moore & Carlson, 2012; Thompson, 1994, 2011). Coordinate systems act as a critical foundation to these ideas because they afford uniting two (or more) quantities in a representational space that permits coordinating and operating on quantities, such as constructing and partitioning directed segment lengths (Lee, 2017; Thompson, Hatfield, Yoon, Joshua, & Byereley, 2017).

Despite coordinate systems providing important affordances that other representations including tables and formulas do not, Lee (2016) argued that researchers and educators have often assumed students have internalized coordinate systems, particularly in research on students' graphing activity. Researchers who have given attention to students' understandings of coordinate systems typically provided students with pre-constructed, conventional coordinate systems with which students were expected to read information from (*e.g.*, Levenberg, 2015) as if the coordinate systems contained some information in-and-of-themselves. Stemming from the exclusive use of conventional, familiar coordinate systems, researchers have provided little attention to how students attend to coordinate systems and construct frames of reference to reason about the coordinate system representations. Notably, mathematics

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education researchers have given more attention to students' understandings of functions and their graphs than the representational space they are constructed upon (*e.g.*, Herscovics, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Schwarz & Hershkowitz, 1999; Oertman, Carlson, & Thompson, 2008; Lloyd, Beckmann, & Cooney, 2010). In these studies, researchers assumed the coordinatized plane as a given structure to be used in constructing or interpreting graphs of functions. We acknowledge that such an assumption is often necessary for practical reasons, much like we assume in the present work that students have constructed interiorized numbers and length as a quantity; a researcher cannot investigate all things at once. We point out, however, that such an assumption leaves much to be understood about how students reason within coordinate systems and associated frames of reference.

By frames of reference, we refer to a mental structure through which an individual situates a quantity. This structure is constructed through the processes of committing to a reference point, a unit of measure, and directionality of measure comparison (*e.g.*, committing to a direction of additive comparison; Joshua, Musgrave, Hatfield, & Thompson, 2015). By coordinate systems, we mean a mental system of coordinated measurements obtained through coordinating multiple frames of reference. Because a coordinate system (and associated frames of reference) is a mental construction, it is idiosyncratic to the individual who has constructed it. Hence, and as we illustrate throughout this paper, the coordinate system a student conceives can differ in marked ways when compared to the coordinate system intended by a researcher or educator.

Addressing the aforementioned gap in research, we discuss a secondary pre-service mathematics teacher's (PST's) graphing activities during a teaching experiment (Steffe & Thompson, 2000) with attention to how she reasoned quantitatively within frames of reference. We use the term *graphing* to include both constructing graphs and determining whether a researcher-presented-graph represents the appropriate relationship between two quantities constituting a dynamic situation or as indicated by a formula. In this paper, we focus primarily on the mental actions involved (*i.e.*, frames of reference) when the PST made inferences about covarying quantities or a graphical representation of covarying quantities. In doing so, we extend the extant literature base in two ways. First, we characterize a PST's constructing and sustaining frames of references when graphing in non-canonical coordinate orientations that were novel to her. Second, we contribute to the available research base by relating Joshua et al.'s (2015) framework of quantitative frames of reference to Piaget's (1970) distinction between *figurative* and *operative aspects of thought.* In relating these frameworks, we describe marked differences in a PST's committing to a reference point and directionality of measure comparison within frames of reference across four graphing tasks.

2. Literature review and theoretical framework

The notion of *frames of reference* appears in multiple areas of study, including spatial cognition, linguistics, physics, and mathematics. In this section, we present an overview of literature on frames of reference from two bodies of research—spatial cognition and quantitative reasoning. We also explain the context in which we study frames of reference and outline a theoretical framework that guides our study. As a second important theoretical framework that guides our study, we elaborate on Piagetian notions of figurative and operative aspects of thought. Finally, we make connections between the two theoretical frameworks we use to describe a student's frames of reference and discuss how we extend both.

2.1. Frames of reference literature in spatial contexts

How people perceive, organize, and represent space through various modalities such as vision, touch, gesture, and language is a central question in spatial cognition research. Spatial cognitionist Levinson (2003) explained that the notion of frame of reference is essential in the study of this question. Borrowing Rock's (1992) definition and crediting the Gestalt theories of perception from which the modern interpretation of frame of reference originated, Levinson defined frame of reference as "[a] unit or organization of units that collectively serve to identify a coordinate system with respect to which certain properties of objects, including the phenomenal self, are gauged" (2003, p. 404).

An emergent theme from the spatial cognition work on frames of reference is that there are multiple ways people can gauge the location of an object in reference to another. Accordingly, spatial cognitionists have conducted numerous experimental studies and provided multiple distinctions between different types of frames of reference used for characterizing spatial relationships among objects. For instance, several researchers have adopted distinctions between three types of frames of reference which differ in their center: the viewer, environment, or object (*e.g.*, Carlson-Radvansky & Irwin, 1993; Farah, Brunn, Wong, Wallace, & Carpenter, 1990).

In a *viewer-centered frame of reference*, objects are represented relative to the viewer's perspective, centered at the perceiver's retina, or head, or body. In an *environment-centered frame of reference*, objects are represented relative to salient features of the environment, "such as gravity or prominent visual landmarks" (Carlson-Radvansky & Irwin, 1993, p. 224). In an *object-centered frame of reference*, objects are represented with respect to an object and the axes intrinsic to the object. Carlson-Radvansky and Irwin (1993) provided an example of these different frames of reference used in describing objects in space, as illustrated in Fig. 1.

In their description of the drawing, Carlson-Radvansky and Irwin (1993) stated:

> Which object is "above" the trash can? From the perspective of the person lying on the couch, object 1 is above the trash can with respect to a viewer-centered reference frame, object 2 is above with respect to an object-centered reference frame, and object 3 is above with respect to an environment-centered reference frame. (p. 225).

Studies by spatial cognitionists identify general features of frames of reference and afford ways to categorize them, yet there are two notable and related elements that we find problematic. First, an explicit account of the perceiver of the situation is missing, and thus it is unclear from whose perspective—the subject or the observer—the frames of reference are held and coordinated. Second, the

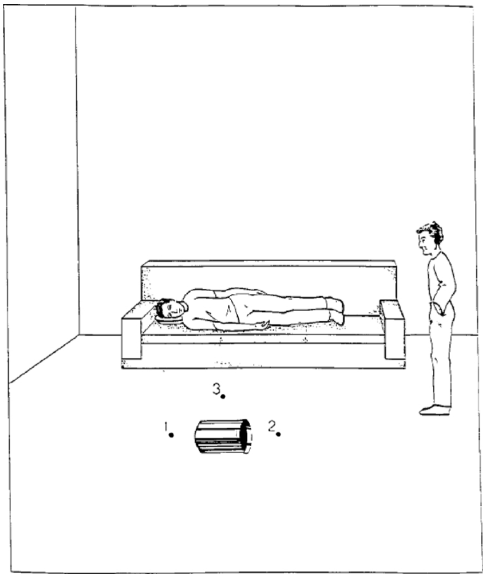


Fig. 1. Fig. 1 from Carlson-Radvansky and Irwin (1993, p. 225).

environment or object that serves for the center of the frame of reference is taken as an objective, stable entity independent of the perceiver; the "top" of the trash can is taken as a property of the can as opposed to a property to be constructed by a perceiver. Taken together, these elements downplay the active role of a perceiver by leaving the ontology of frames of reference unclear.

To further illustrate, consider the example in Carlson-Radvansky and Irwin (1993). Although subtle, it is unclear whether Carlson-Radvansky and Irwin consider these frames of reference from an observer's perspective or impute these frames of reference to the person lying on the couch; there is a difference between claiming "object 1 is above the trash can with respect to a viewer-centered frame" and saying "if the person lying on the couch says that object 1 is above the trash can, then he is describing the location of the object with respect to a viewer-centered frame of reference." The former claim leaves the perceiver implicit and the latter claim makes clear that it is the observer's attempt to explain how a subject is perceiving the situation. A similar analogy could be made for objects 2 and 3 in the situation. Notably, and with respect to objects 2 and 3, for the subject to perceive those as "above" the trash can, it requires that the subject decenter from his viewer-centered reference frame in order to establish a frame of reference distinct from his current view.

Decentering (Piaget & Inhelder, 1967) requires the perceiver to be aware of his or her current perspective but also be able to disembed (Steffe & Olive, 2010) himself from that perspective in order to adopt a perspective of another. Like spatial cognitionists, Piaget and colleagues found sensory input to be an important source for constructions of space. However, in contrast to the notion of frame of reference in spatial cognition literature, Piaget and colleagues developed frame of reference around the active perceiver (*e.g.*, child or student) in order to explain how a subject constructs, perceives, and operationalizes space. That is, and aligning with our focus here, Piaget and his colleagues emphasized the importance of investigating the process by which the human mind constructs, abstracts, and operationalizes space from sensory foundations and through active (logico-mathematical) mental actions that enable the perceiver to co-ordinate and transition between different frames of reference (*e.g.*, decentering). We consider the perspective of the perceiver and his or her ability to decenter from his or her viewer-centered frame important because coordinate systems entail multiple frames of reference, at least one of which is not likely viewer-centered (Lee, 2017).

2.2. Frames of reference in quantitative reasoning

Consistent with the Piagetian emphasis on the active construction process of the human mind, Thompson (2011) highlighted the important role of an individual constructing and reasoning about *quantities*. A quantity is a conceptual entity an individual constructs as a measurable attribute of an object. *Quantitative reasoning* refers to conceiving of and reasoning about quantities and relationships between quantities that arise from quantitative operations (Smith & Thompson, 2008; Thompson, 2011). Building on Thompson's (2011) description of quantitative reasoning, Joshua et al. (2015) illustrated that in the extant literature on reasoning about quantities within frames of reference, and compatible with our critique of the aforementioned spatial cognition work, researchers treat frames of reference as "objects external to a person reasoning with it" (p. 31) or "defined by the existence of a concrete object" (p. 36). In response to this observation, Joshua et al. emphasized the importance of focusing on the mental activity and the cognitive processes of constructing and sustaining frames of reference.

Joshua et al. (2015) defined a *frame of reference* as "a set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 2) and offered a model of mental actions involved in conceptualizing measurable attributes within a frame of reference:

> An individual conceives of measures as existing within a frame of reference if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives meaning to a zero measure and all non-zero measures, and 3) committing to a directionality of measure comparison additively, multiplicatively, or both. (p. 32).

According to Joshua et al. (2015), when an individual works within *one* frame of reference, she works consistently with the same reference point, unit of measure, and directionality of measure comparison.

Joshua et al. (2015) differentiated frames of reference from coordinate systems by making a distinction between conceptualizing frames of reference as mental activity and a coordinate system as the *product* of the mental activity involved in conceptualizing multiple frames of reference and coordinating those frames of reference. This product "allows us (mathematicians, teachers, and students) to represent the measures of different quantities simultaneously when those measures stem from potentially different frames of reference" (Joshua et al., 2015, p. 35). Saldanha and Thompson (1998) described the activity of holding in mind the measures of different quantities simultaneously, permitted by a coordinate system, as forming a *multiplicative object*. More recently, Thompson et al. (2017) described this process to entail uniting *quantities' values.* Further, in addition to uniting quantities' values, Thompson et al. emphasized the importance of constructing a multiplicative object of quantities' values such that the unity is sustained as an individual tracks the quantities' values as they *vary* in tandem in a coordinate system.

In this paper we discuss a PST's reasoning about quantitative relationships and their graphical representations in non-canonical coordinate systems by describing how she constructs quantities within frames of reference and how she coordinates multiple frames of reference to reason about the relationships (if any) between two quantities. When discussing the PST's reasoning within frames of reference, we narrow in on *committing to a reference point* and *directionality of measure comparison* among the three mental actions involved in Joshua et al.'s (2015) framework. There are three primary reasons in doing so. First, and although committing to a unit of measure is an equally important element of reasoning within frames of reference, the other two mental actions are most relevant to characterizing evolutions in the PST's activities. Second, and as we describe in more detail below, those two mental actions align with our focus on non-canonical coordinate system orientations and associated frames of reference. Coordinating activity among differing coordinate system orientations requires one to operationalize reference points and directionality of measures. Third, when engaging in graphing activities that involve coordinating two quantities and determining the quantitative relationship between the two quantities, it is critical to attend to how one quantity varies in relation to the other (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). In order to account for variation in quantities, a reference point and directionality of measure comparison are necessary, but it is not necessary for an individual to explicitly construct and sustain a unit of measure. Instead, an individual can reason about quantitative magnitudes and gross comparisons between these magnitudes (Castillo-Garsow, 2012; Saldanha & Thompson, 1998; Thompson & Carlson, 2017).

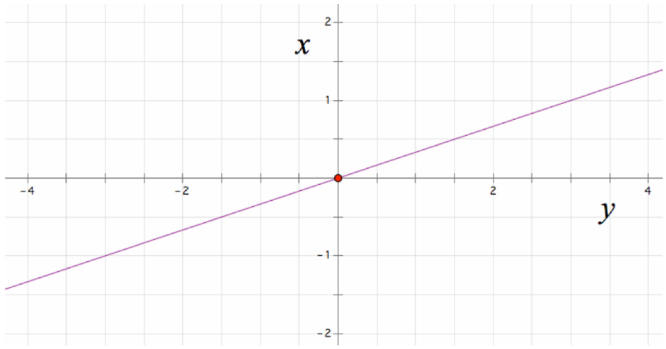
2.3. Figurative and operative thought

To investigate and characterize one's committing to a reference point and committing to a directionality of measure comparison, we extend Joshua et al.'s (2015) description using Piaget's (1970) distinction between figurative and operative aspects of thought. According to Piaget, *figurative thinking* is based in, constrained to, and dominated by perceptual elements and sensorimotor experience (Montangero & Maurice-Naville, 1997). Therefore, as Piaget (1970) explained, "The figurative aspect [of thought] is an imitation of states taken as momentary and static" (p.14). On the other hand, Piaget (1970) explained, "The operative aspect of thought deals not with states but with transformations from one state to another" (p. 14). As such, *operative thinking* foregrounds the coordination and re-presentation of mental actions and the transformation of those actions so that figurative material (including sensorimotor experience) is subordinate to the coordination of mental actions (Montangero & Maurice-Naville, 1997).

Moore and Thompson (2015) and Moore (2016) adopted these notions to develop models of students' graphing activity, and in doing so provided accounts for students' graphing activities dominated by either figurative or operative aspects of thought (*i.e.*, static or emergent shape thinking, respectively). For students whose graphing activities are dominated by figurative aspects of thought, student actions are subordinate to perceptual (figurative) properties of shape including perceived sensorimotor experiences like movement. Hence, these students tend to operate on a graph as an object in-and-of-itself and their thinking is based on perceptual cues or physical features of a graph. On the other hand, students whose graphing activities are dominated by operative thought are not constrained to perceptual cues or sensorimotor experience, but instead attend to coordinations of actions (*e.g.*, quantitative relationships). These students tend to understand graphs as coordinated traces, "with the trace being a record of the relationship between covarying quantities" oriented within a coordinate system (Moore & Thompson, 2015, p. 785).

Moore and Thompson (2015) illustrated these differences in students' graphing activities using the graph and prompt in Fig. 2. As shown in Fig. 2, a hypothetical middle school student's graph of $y = 3x$ was presented on what the authors designed to be a non-canonically oriented coordinate system with the horizontal and vertical axes each representing y and x, respectively. They first described a student, Student 1, rotating the graph 90° counterclockwise to orient the coordinate system with the x-axis positioned horizontally and determined that the middle-schooler's graph was incorrect because the line falls downward left-to-right and, hence, the "slope" was negative. They then described a second student, Student 2, examining the graph as given and then in a rotated orientation (*i.e.*, Fig. 2 rotated 90° counterclockwise) while sustaining an image of how one quantity changed in relation to the other. The second student explained, "Change in y is three times change in x" (p. 783) regardless of orientation.

Moore and Thompson (2015) argued that Student 1 and Student 2 demonstrate two different ways of thinking in their graphing activity; the former engaged predominantly in figurative thinking and the latter predominantly in operative thinking. Student 1 drew



A middle-school student graphed the relation defined by y=3x as shown in Figure 1. How might he/she have been thinking when producing the graph?

Fig. 2. Fig. 1 and prompt in Moore and Thompson (2015, p. 783).

upon perceptual cues of the graph and associated properties of a slope with a line's direction or shape. Student 2 attended to the images of covarying quantities and thus was not constrained to the orientation of the graph or perceptual cues. Although making a useful distinction between figurative and operative aspects of thought in explaining students' graphing activity, we note that Moore and Thompson (2015) and Moore (2016) did not give explicit attention to frames of reference; we point out that Student 2's actions necessarily rest on her coordinating and transforming frames of reference in order to claim that the given graph, in both orientations, represented an invariant covariational relationship. Although we draw on Moore and Thompson (2015) and Moore (2016) work in using figurative and operative distinctions to describe a PST's graphing activity, we extend this work by giving explicit attention to a student's understanding of frames of reference and coordinate systems in such instances.

3. Methodology

The data we present and analyze is from a teaching experiment (Steffe & Thompson, 2000) conducted over the course of a semester at a large public university in the southeastern U.S. with three PSTs, one of which was Lydia. Our goal in the teaching experiment was to investigate Lydia's ways of thinking and create models of her quantitative reasoning. Specifically, as we describe here, we explored Lydia's reasoning about relationships between quantities in the context of graphing including how she established frames of reference in associated coordinate systems.

3.1. Participants and setting

At the time of the study, Lydia was a junior undergraduate student enrolled in both a content and pedagogy course. Lydia had completed at least two additional courses beyond a traditional calculus sequence with at least a C as her final grade in each course. Lydia volunteered to participate in the study and received monetary compensation for her time. In addition to considering her availability, we selected Lydia from a pool of PST volunteers because she provided a range of responses and communicated her thinking clearly in written responses to questions related to rate of change, interpretation of graphs, symbolic notation, and proportion in an adapted version of the *Aspire* instrument (Thompson, 2016). In this paper, we focus on Lydia—as opposed to all three PSTs who participated in the teaching experiment—because we observed a notable transition in her reasoning within frames of reference, and particularly her commitment to a reference point and directionality of measure comparison.

Lydia participated in 12 videotaped interview and teaching sessions. Each session lasted approximately 1–2 h in length. In each session, a research team member served as the teacher-researcher (TR), who interacted with Lydia and sought to engender and model changes in her ways of thinking during the teaching sessions. In this process, the TR engaged in developing and testing on-going models of Lydia's current ways of thinking. Other research team members acted as witness-researchers (WR), who managed the video camera, asked follow-up questions the TR did not ask, and provided alternative explanations of Lydia's activities during research meetings (Steffe & Thompson, 2000).

Our overall goal of the teaching experiment in which Lydia participated was exploring the mental actions involved in constructing abstracted quantitative structures such as the covariational relationships associated with sine or cosine. An abstracted quantitative structure is "a quantitative structure that becomes so internalized and operational that [a person is] able to assimilate novel representational contexts and situations to that structure" (Moore & Silverman, 2015, p. 523). Based on this goal and previous work exploring students' representational activity with multiple coordinate systems (Moore, Paoletti, & Musgrave, 2013; Moore, Silverman, Paoletti, & LaForest, 2014), our initial image of the teaching experiment sessions involved engaging Lydia in a multitude of contexts that afforded her conceiving invariant quantitative structures despite what we perceive to be perceptual or figurative differences in these contexts. Specifically, we engaged her in a series of tasks—see the Results section for a subset of the tasks—that involved exploring dynamic events, exploring dynamic geometry environments with covarying segments, and graphing in multiple coordinate system orientations for the purpose of inferring the mental actions involved in a student repeatedly constructing a quantitative structure (*i.e.*, the sine relationship, cosine relationship, or a linear relationship) and increasingly differentiating attributes of that structure from figurative material and attributes specific to each context (*i.e.*, a graph of a relationship in one coordinate orientation is perceptually different than a graph of a relationship in an alternative coordinate orientation). We note that although we anticipated that frames of reference is critical to a student constructing an abstracted quantitative structure, we did not anticipate or formulate a priori the results discussed in this paper. Rather, the results we present emerged from our sustained interactions with Lydia and our attempt to provide a viable explanation of her activity in the form of a model of her thinking.

3.2. Data collection and analysis methods

During the 12 sessions, the research team collected video recordings from two cameras, Lydia's written work, and notes taken by research members who were present. There were two video cameras; one captured a wide-angle view facing the student from the front and the other captured a focused view of the students' activities from above. The research team also used a screen recording program to capture a tablet device displaying animations as necessary. The research team constructed annotated transcripts and digitized Lydia's written work for analysis.

We conducted both on-going and retrospective conceptual analyses (Steffe & Thompson, 2000; Thompson, 2008). In on-going analyses, the research team tested and formulated new hypotheses of student thinking throughout the teaching sessions based on the ways Lydia engaged in the tasks. Specific to this manuscript, during on-going analysis, we inferred that Lydia experienced perturbations when graphing quantitative relationships in non-canonical orientations of coordinate systems, with these perturbations apparently stemming from perceptual differences created by graphing in different orientations. These observations led us to modifying subsequent sessions to further investigate this issue including what reorganizations might support Lydia in reconciling her experienced perturbations.

In retrospective analysis of data, the research team revisited the data after completion of the teaching experiment in order to build and revise working models of Lydia's reasoning. The retrospective analysis involved our building tentative models based on Lydia's moment-to-moment current ways of thinking and then testing those tentative models by finding supporting or contradicting evidence from Lydia's activities throughout the teaching experiment. Through this process, we either rejected or generated models of Lydia's shifts in her reasoning over the course of the teaching experiment. In our retrospective analysis pertaining to what we discuss in this paper, we hypothesized that Lydia's repeated opportunities to differentiate attributes of a quantitative structure from its figurative attributes specific to a particular context or coordinate system orientation may have supported shifts in her coordination of frames of reference. We conducted in-depth analysis of Lydia's activity graphing relationships between two quantities with more focused attention to the interplay of figurative or operative thought in her committing to a reference point and directionality of measure comparison. In reviewing the data, and in conjunction with reflections on our on-going analysis, we identified instances that would offer us insights into Lydia's graphing activity and performed a conceptual analysis (Thompson, 2008) with these constructs in mind. For instance, for our current purposes, we identified instances in which we perceived Lydia to experience the need of establishing a frame of reference, regardless of the nature of that frame of reference. It is important to note that our explanatory models are inferences based on Lydia's observable activities.

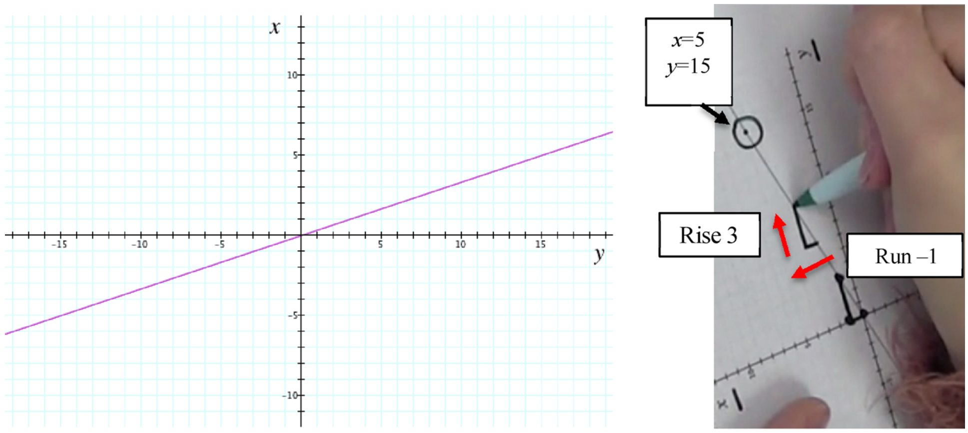
4. Results

In this section we analyze Lydia's activities during four tasks—Tasks A, B, C, and D—from the teaching experiment to illustrate the ways she established and maintained frames of reference within the context of interpreting quantitative relationships (if any) in dynamic situations and graphing. We first provide two instances of Lydia's graphing activity that heavily relied on figurative or sensorimotor elements when committing to a reference point and directionality of measure comparison. Next, we present Lydia's engagement in a task to illustrate her transition from relying primarily on figurative aspects to thinking operationally within frames of reference. Finally, we share an instance that evidences Lydia's committing to a reference point and directionality of measure comparison dominated by operative aspects of thought.

4.1. Lydia's committing to a reference point and directionality and figurative thought

For Task A, we adopted the task from Moore and Thompson (2015) discussed in Fig. 2 (see Fig. 3a for modification). We presented Task A to Lydia previous to any intervention in the teaching experiment to infer her initial ways of reasoning about quantitative relationships represented on a non-canonical coordinate system. Similar to Student 1 in the example provided above (section 2.3; Moore & Thompson, 2015), Lydia rotated the paper counterclockwise 90° such that the x-axis was horizontal from her perspective as shown in Fig. 3b. She then concluded the "slope" of the line was "rising negative three." To illustrate, Lydia picked two points on the line and drew a horizontal line segment left one grid mark and a vertical line segment up three grid marks, connecting her two points (Fig. 3b). Lydia explained, "If I were to rise here…I'm rising this three [*referring to the vertical line segment*]…and then I'm running negative one [*referring to the horizontal line segment*], which would then [be] three over negative one x [*writes* '$\frac{3}{-1}x$'] still equals negative three x [*writes* '$= -3x = y$']."

Noticing that Lydia associated moving to the left with "running negative one," the TR asked Lydia to provide coordinates for a circled point on the line (Fig. 3b). Reading off the abscissa and ordinate of the circled point from each axis, Lydia determined $x = 5$



a) b)

Fig. 3. (a) Task A graph (b) Lydia solving Task A.

and $y = 15$. We interpreted this to indicate that Lydia was aware that the positive numbers for x were to the left of the vertical axis and that the positive numbers for y were above the horizontal axis. In other words, Lydia was aware of the quadrant in which she was working entailed positive values for both x and y, compatible with the graph as designed by the research team.

Despite recognizing each coordinate value in the particular quadrant to be positive, Lydia maintained associating moving horizontally to the left or right with a decrease or increase, respectively, in the quantity's value. We inferred from her activity that Lydia's commitment to directionality of measure comparison was associated with sensorimotor activity. Specifically, moving to the left one grid mark meant decreasing in value by 1, even though she was aware that left to the vertical axis entailed positive x-coordinates. We note that this commitment was inconsistent with the coordinate plane representation upon which *we* had constructed the student's graph. In other words, although Lydia did commit to directionality, this directionality was different from committing to the quantitative organization as we conceived the axes to represent.

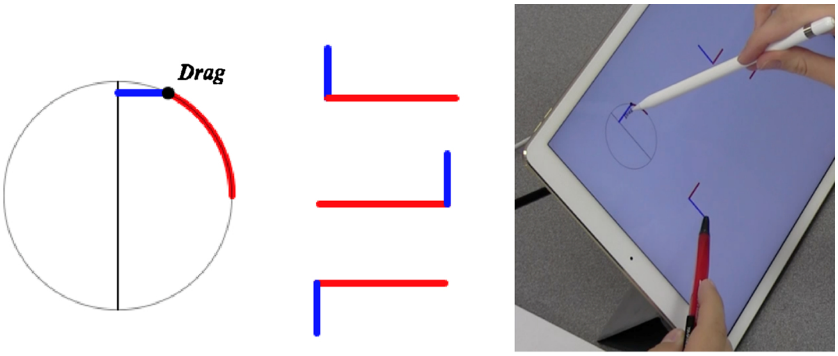
Finding the coordinates of a single point, $x = 5$, $y = 15$, brought Lydia's attention to the inconsistency between the equation she determined, $y = -3x$, and the point on the graph she identified. She understood that $y = -3x$ was not accurate for the paired values, while $y = 3x$ was consistent with the particular paired values but did not capture the "slope" of -3 as indicated by her movements. This inconsistency evidently perturbed her thinking, as Lydia claimed, "This is so annoying," and she was unable to reconcile these inconsistencies leaving her perturbed over the last 15 min of the task. As such, Lydia was able to unite quantities' measures for a single point; however, when it came to uniting quantities' measures as they were changing to determine the "slope" of the line, her commitment to directionality of measure comparison of each quantity was dominated by figurative aspects of thought.

In a subsequent task, Task B, we intended to explore what Lydia constructed and sustained as a quantitative structure (*e.g.*, the sine relationship) without pre-constructed axes or numerical values other than varying segment magnitudes. In the task, we provided Lydia with a circle situation in a Dynamic Geometry Environment (DGE) in which she was able to drag a point along a circle. As Lydia dragged the point on the circle counterclockwise, two quantities were highlighted with different colors; the arc length starting at 3 o'clock in red and the horizontal distance to the right of the vertical diameter of the circle[1] in blue (see left in Fig. 4a). Also displayed in the DGE were three different representations of the two quantities (see right in Fig. 4a). In each representation, we arranged two segments in a rectilinear manner; one red segment equivalent in length to the arc highlighted on the circle and one blue segment equivalent in length as the horizontal distance highlighted in the circle. The lengths of the red and blue segments varied as the point moved along the circle (see https://youtu.be/NjKt2KWLtuo for a video illustrating their variation). The red segment was always oriented horizontally and the blue segment was always oriented vertically. The corresponding axes and reference points for these segments were hidden in the DGE in order to minimize the perceptual material available.

We designed all three segment-pairs to covary appropriately in length with the only difference being the location of the hidden fixed points. Fig. 5 presents the bottom segment-pair (in Fig. 4a) in five different positions associated with instantiations of the variable point on the circle. A video demonstrating this segment pair as the point on the circle moves can be found at https://youtu.be/LGC9kKBBmGk. As illustrated in Fig. 5, we designed the bottom segment-pair such that the red segment emanated to the right of the blue segment as the two segments covaried. The pair was such that the fixed reference point was the endpoint of the blue segment *not* intersecting with the red segment. This meant that the blue segment emanated up and down of the fixed point as the horizontal distance from the circle center increased or decreased, respectively.

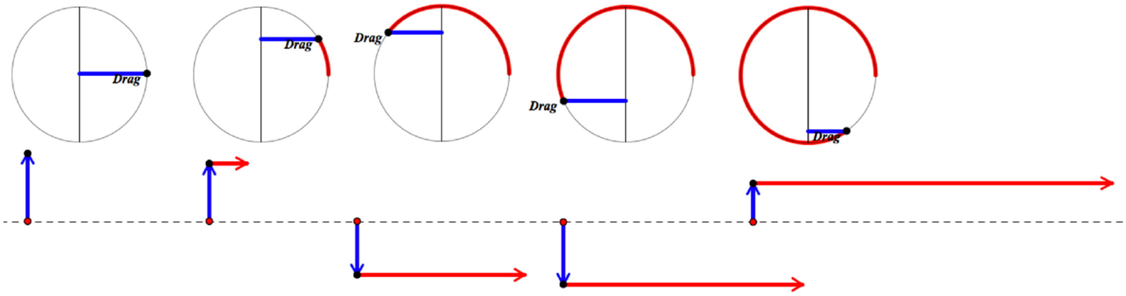
To begin the task, we asked Lydia to determine which, if any or all, of the three representations accurately captured the relationship among the two quantities (*i.e.*, the arc length and horizontal distance to the right of the circle center) as the point was dragged along the circle. We present Lydia's activity with the segment-pair illustrated in Fig. 5, which is representative of her activity

[1] In previous tasks (constructing the sine and cosine graph), Lydia perceived these quantities as directed distances.



a) b)

Fig. 4. Task B in a DGE. (a) The circle and three representations of the quantities' magnitudes. (b) Lydia drags the blue segment on the circle as the TR places pen at the end of the corresponding blue segment (and hidden fixed point) of what was the bottom segment-pair in Fig. 4a (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article).



a) b) c) d) e)

Fig. 5. The bottom segment-pair in five different positions of the point on circle. The dash-line and arrows are added to indicate the direction of variation for each segment as the point moves counterclockwise (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article).

throughout the task. During the task, the TR positioned the segment-pair below the circle as in Fig. 4b to isolate it as Lydia determined whether it accurately captured the relationship among the two quantities. Next, Lydia dragged the point along the circle into various positions and attended to the length of each segment in the segment-pair. When reasoning about how the quantities (horizontal distance and arc length) changed in tandem, she attended to the change in lengths of each segment in the segment-pair. Looking at the segment-pair, Lydia assigned negative and positive lengths to the blue segment (*i.e.*, horizontal distance) based on its relative position to the red segment—she conceived that above the red segment implied positive blue segment lengths and below the red segment implied negative blue segment lengths. For example, in Fig. 5, Lydia considered the blue segment lengths in (b) and (e) as negative and those in (c) and (d) as positive.[2]

From this we inferred that Lydia established and committed to a reference point being the point of intersection of the blue and red segments. Her commitment to directionality of measure comparison was based on that reference point, below the intersection point meaning negative and above meaning positive. However, by design of the pair she was considering, Lydia's reference point moved and thus, her frame of reference varied as the segment lengths varied. Not enacted yet was a coordination of these multiple frames of reference with a commitment to a *fixed* reference point from which the directionality of measure comparison could be gauged despite the relative positions of the two segments.

Noticing that Lydia was focusing on the physical positions of the segments with respect to each other and not necessarily to a fixed reference point, the TR placed his pen at the fixed endpoint of the blue segment (see red pen in Fig. 4b and red circle in https://youtu.be/LGC9kKBBmGk) to see if Lydia altered her attention to a fixed reference point. *Excerpt 1* starts with Lydia's response to the TR's action and her perturbation by the segment's behavior.

Excerpt 1. Lydia makes sense of the bottom segment-pair in Fig. 4a.

[2] We note that Lydia maintained compatible commitments with each segment-pair.

Lydia: [*Drags the point on the circle counterclockwise from the initial position, looking at the segment-pair while the TR has his pen placed at the fixed point* (Fig. 4b).] So the arc length is increasing, and the [*pauses for two seconds*] I don't like what you're showing me right now.

TR: Why don't you like what I'm showing you right now?

Lydia: [*Continuing to drag the point on the circle into the second quadrant*] Because the bottom of the blue isn't moving, just the top, like the top of it [*referring to where the blue segment intersects the red segment on the segment-pair*]. [*Returns point on circle to the first quadrant*] So I don't know if that changes everything [*Places her hand on her forehead and expresses confusion and disappointment*]. Okay, so it's [*referring to the blue segment of the segment-pair*] decreasing [*referring to the blue segment in the circle, dragging the point counterclockwise on the circle in the first quadrant*], becoming more negative [*referring to the blue segment of the segment-pair*], and now it's [*referring to the blue segment of the segment-pair, dragging the point on the circle counterclockwise in the second quadrant*] increasing, but it's [*referring to the blue segment of the segment-pair*] positive. Now it's [*referring to the blue segment of the segment-pair, dragging the point on the circle counterclockwise in the third quadrant*] decreasing, becoming less negative. No, [*continuing to drag the point on the circle counterclockwise in the third quadrant*] decreasing, becoming less positive.[*Sighs*] Hmm.

TR: What are you thinking?

Lydia: So then that one [*referring to the blue segment in the segment-pair when the point on the circle is in the second and third quadrant*] doesn't show us the negative distance [*e.g., she understands the blue segment in the circle is negative when the point is in the second or third quadrant*].

TR: How’s this one [*pointing to the blue segment in the segment-pair such that the fixed point is at the beginning point for each segment,i.e., the top segment-pair in* https://youtu.be/NjKt2KWLtuo] show that it's a negative distance?

Lydia: Because it's [*referring to the blue segment of the segment-pair*] under the red line. Are you going to do the pen thing [*referring to placing the pen at the fixed endpoint* (Fig. 4b)] again and change my whole [*the TR replaces his pen at the fixed point* (Fig. 4b), *returning her attention back to what was the bottom segment-pair*]. Okay, so this [*referring to what was the bottom segment-pair*] is saying because the red line is below the blue that the [*moving the point counterclockwise in the second quadrant*] we're increasing in *x* distance as the arc length is increasing. But it should be a negative, it should be, essentially decreasing more because it's [*referring to the blue segment on the circle*] supposed to be a negative distance.

TR: Becoming more negative.

Lydia: Yes.

TR: Because this [*referring to the blue segment of the segment-pair*] is above the red line.

Lydia: [*Interrupting*] It's [*referring to the blue segment of the segment-pair*] showing a positive distance.

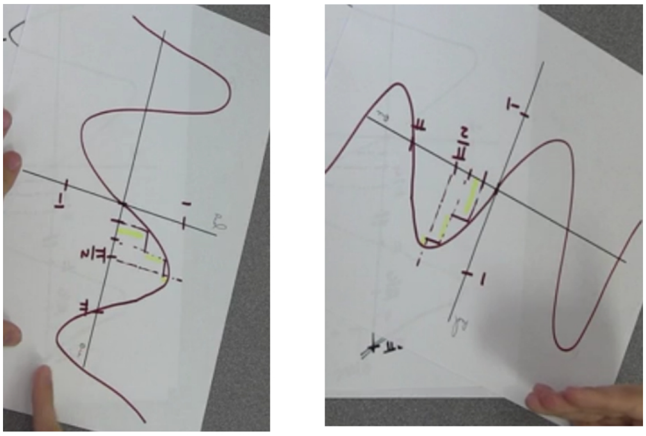
As implied in her comment “I don’t like what you’re showing me right now,” Lydia was perturbed by her observation that the point TR was pointing to did not move. As Lydia considered the segment-pair, she briefly used the fixed reference point implied by the TR’s pen to describe the variation in the blue segment’s length (*e.g.*, “decreasing, becoming less negative” as the point moves counterclockwise in the third quadrant). However, this recognition was not immediate for Lydia and her reasoning about the accumulation of the quantity within her frame of reference with the fixed reference point was brief as demonstrated in *Excerpt 1*. Lydia returned to and sustained using the red segment, regardless of which segment-pair was under consideration, as a reference to gauge the sign and variation of the blue segment’s length. In other words, Lydia worked predominantly within her initial frames of reference with a moving reference point and a commitment to directionality of measure comparison based on the physical location of the blue segment in relation to the red segment.

Looking across both Tasks A and B, we inferred Lydia’s reasoning within quantitative frames of reference to indicate figurative aspects of thought in the form of sensorimotor elements. For example, moving to her left and right necessarily implied a decrease and increase in values, respectively. Or, the position of a segment above and below another segment necessarily implied positive and negative magnitudes, respectively. We refer to such frames of reference as *figurative frames of reference*. To be clear, a figurative frame of reference is a frame of reference in that it serves as a mental structure used to situate quantities (or some form of) with a commitment to a reference point and some directionality of measure comparison. But, the commitment to a reference point or some directionality of measure comparison is a figurative commitment.

4.2. Rotating a graph and reconsidering directionality of measure comparison

Based on on-going analysis in which we inferred Lydia engaging in figurative elements specific to each context, we designed a sequence of tasks to have her consider graphing in various Cartesian coordinate system orientations. In Task C, one of the latter tasks in this sequence, Lydia revisited a task in which she had constructed the sine graph in a canonical Cartesian coordinate system orientation, with the horizontal axis representing a directed arc length along a circle and the vertical axis representing the vertical distance above the horizontal diameter of a circle. We asked her to determine if that graph, when rotated (including axes) in different orientations, still represents the appropriate relationship of how arc length and vertical distance change together. Our goal was for her to divorce physical or sensorimotor elements from her construction of a quantitative structure (*i.e.*, the sine relationship) in order to conceive that the graph, no matter how rotated, represents the sine relationship.

Fig. 6 displays two of those orientations positioned so that the reader’s vantage point is aligned with Lydia’s. For example, Fig. 6a shows Lydia’s initial sine graph rotated 90° clockwise. The arc length axis is positioned vertically with respect to the viewer, with positive values oriented downward; the vertical distance axis appears horizontally with positive values oriented to the right. *Excerpt* 2 provides Lydia’s response to the task beginning with the initial rotation to 90° clockwise (Fig. 6a).



a) b)

Fig. 6. Lydia's graph—the canonical sine graph—(a) rotated 90° clockwise and (b) rotated 180° clockwise.

Excerpt 2. Lydia reasons about her sine graph in different orientations.

Lydia: [*Rotates paper* 90° *clockwise, see* Fig. 6a] Yes. And then here we're increasing in arc length [*moving finger from origin down along the arc length axis*]. We're having a positive distance in arc length, having a positive distance in height [*moving her finger from origin to the right along the height axis*], so yes, yes.

TR: Now so you're going to turn it, right? One more. Are we guaranteed it's going to maintain the relationship in the next turn or is that just a conjecture at this point?

Lydia: Conjecture.

TR: Okay.

Lydia: Because I haven't really looked at it. [*rotates paper another* 90° *clockwise, see* Fig. 6b] Okay, so here we have a positive distance in arc length [*motions from origin left along the arc length axis*] from zero to pi over two and pi over two to pi, and then we have the positive distance in our height [*motions hand from horizontal axis down*], and it reaches a maximum of one, negative one [*lays hand horizontally at first peak right of vertical in* Fig. 6b *axis*], so yes.

TR: Mm-hm, now we have another rotation we can do.

Lydia: [*Attempts to rotate the graph, but TR stops her.*] Okay. [*laughs*] Got a little eager.

TR: Before we do that are we—is it still just a conjecture that it's going to maintain the relationship or can we absolutely positive that we're going to have the appropriate relationship? So not actually positive and we've seen a pattern for three out of the four, you know, is it still just a conjecture, like it can maybe not work, or is it like [*inaudible*] has to work when we rotate it.

Lydia: If I'm being skeptical, I don't want to say it's going to get guarantee that it's going to follow the pattern unless I rotate it and like can visualize.

As demonstrated in *Excerpt* 2, and differing from her actions on Tasks A and B, Lydia attended to a reference point and direction of change in quantities' measures aligned with the origin and the axes of the coordinate system, respectively. In doing so, she coordinated increases and decreases in those quantities in both orientations of the coordinate system and accompanying sine graph. Importantly, her attention to direction of change in length (*i.e.*, commitment to directionality of measure comparison) was divorced from certain perceptual or physical cues (*e.g.*, movements upward did not necessarily imply increasing values and the positioning of a length above another length did not necessarily imply positive values). Lydia was able to transform her frames of reference in order to account for figurative differences in magnitude orientations. From this we infer that Lydia has shifted from committing to figurative aspects to committing to the quantitative organization associated with the quantities' measures represented along axes. In this sense, we claim Lydia showed signs of establishing *operative frames of reference*, which were based on a stable reference point and directionality of measure comparison.

As the teaching session moved forward, Lydia continued to verify if graphs, when rotated, represented the same quantitative relationship in different orientations by purposefully and sequentially sweeping her finger and hand along the axes. When the TR asked her to predict each case of rotation, Lydia was reluctant to conclude that the relationship would be maintained before carrying out the physical activity of rotating the paper and moving her fingers along the axes. We interpret this to mean that it was necessary for Lydia to carry out sensorimotor activity and instantiate each case using the perceptually available material when the physical orientation of the axes changed. Through such activity, Lydia re-established reference points and directionality of measure comparison for each instantiation of the graph orientation. For each instantiation of the graph orientation, Lydia showed signs of establishing operative frames of reference; however, she was yet to anticipate an invariant relationship across all orientations and associated frames of reference.

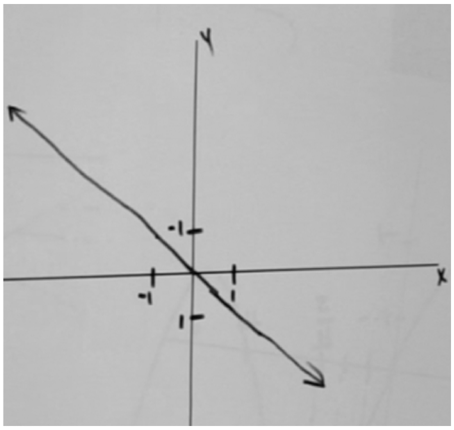


Fig. 7. Graph of $y = x$ on a non-canonical coordinate system.

4.3. Lydia's committing to directionality supported by operative thought

After Task C, we had Lydia consider Task D, a variation of Task A, in order to continue exploring her constructing a quantitative structure in different coordinate system orientations but in a different context than the sine relationship. The TR provided Lydia with the graph displayed in Fig. 7, which was intended as the graph of $y = x$ on a non-canonical coordinate system in which y increases as one moves *downward* along a vertical axis and x increases as one moves to the *right* along a horizontal axis.

Using this graph in its given orientation, Lydia initially associated the "slope" of the line and the relationship between the two quantities x and y as demonstrated in *Excerpt 3*.

Excerpt 3. Lydia reasons about "slope" and the relationship between the two quantities x and y in Fig. 7.

Lydia: Okay. Okay, so this is just [*pauses for five seconds*] So if x is one, then y should be one, which this looks like it has a negative slope, but it doesn't have a negative slope. Visually like from what we're used to a negative slope looking like, it doesn't.

TR: So why doesn't it have a negative slope?

Lydia: Because if I rise one [*places her pen at (0, 0) and then moves to the right towards (1, 0)*], then I'm like going down [*motions down to point (1, 1)*], but I'm rising a value, like a positive value [*points to (1, 1) located in the fourth quadrant of non-canonical Cartesian coordinate system*].

After *Excerpt* 3, Lydia rotated the graph 90° clockwise and explained, "we are decreasing at an x value and decreasing at a y value."[3] As the discussion continued, Lydia spoke generally about slope associations. She explained that a positive slope is associated with x and y both decreasing or increasing, and that a negative slope is associated with y increasing or decreasing as x decreases or increases, respectively. In response to Lydia's generalizations, the TR then rotated the paper and graph in Fig. 7 180° (Fig. 8a) and asked Lydia to draw "something that would have negative slope." We present Lydia's response in *Excerpt 4*.

Excerpt 4. Lydia constructs a graph that is associated with "negative slope".

Lydia: Um, so as I decrease in x [*moves from origin to (−1, 0) on x axis*], then my–There should be an increase in y [*moves from (−1, 0) upward to (−1, 1)*], so it would like cross [*motions line that would cross the original line (in black) through origin*] if we followed that pattern.

TR: Maybe use the green and just draw in something or whatever [*Lydia draws in green line* Fig. 8*a*], just something like that would have a negative slope.

Lydia: Mm-hm.

TR: And then if we rotated it around, would it maintain that?

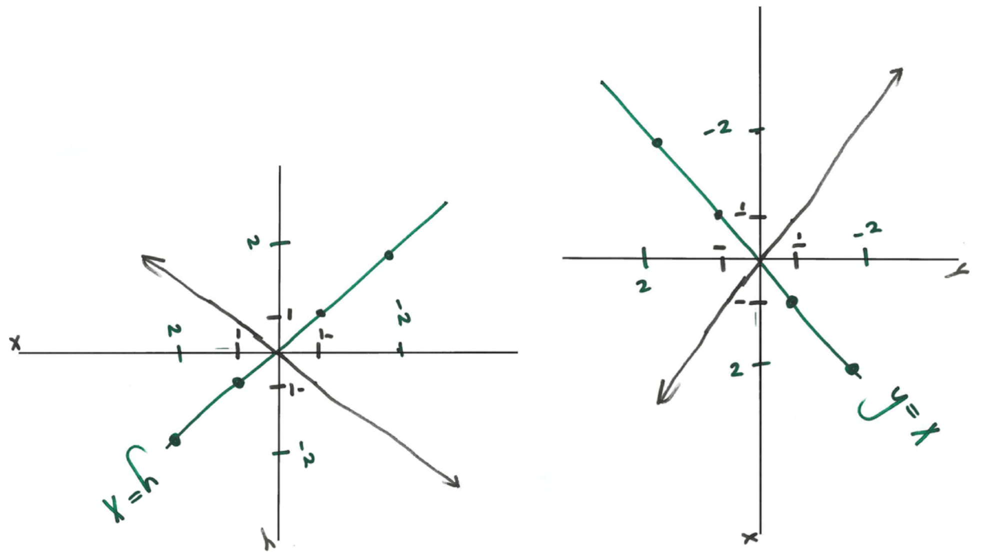
Lydia: Mm-hm. [*Lydia then rotates* Fig. 8a 90° *counterclockwise. See* Fig. 8b]. Of. I don't want to say this opposite relationship, but this positive, negative relationship.

TR: And what's this positive, negative relationship? Say a little bit more.

Lydia: That as we increase one, the other decreases, whether it's x or y. Whereas a positive slope they both increase. They both increase and they both decrease.

[*As the interaction continues, a WR asks her to"show an example of one decreasing and the other increasing" with the graphs displayed as in* Fig. 8b.]

[3] Lydia associated this stated relationship and rotated graph with a "negative slope" because of the "decreasing" relationship. We point out, however, that although she associated her current relationship with a "negative slope", that she maintained a commitment to directionality of measure comparison that was consistent with that intended no matter the rotation of the paper and perceptual orientation of the graph.



a) b)

Fig. 8. Lydia's drawn graphs (a) in the orientation in which she drew her graph (b) in an orientation rotated 90° counterclockwise from that. ($y = x$ is in reference to the given graph) (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article).

Lydia: If one decreases, then we could say we're on the green, yes. So my y is decreasing to negative 1 [*places her pen at the origin and then moves horizontally right to (0, −1)*] and then with this function [*referring to her drawn graph*], my x is increasing to 1 [*places her pen at (0, −1) and then moves perpendicular to (1, −1), plots (1, −1)*], so even though it's really confusing me with all the rotation. Okay, yes. Or if we were to increase in y [*moves horizontally left to (0, −1) on y-axis*], then on this point, then we would decrease our x to negative 1 [*moves perpendicular to (−1, 0) on x-axis, plots (−1,1)*].

In *Excerpt 4*, when the TR asked Lydia if the graph would maintain the relationship if they were to rotate it, Lydia immediately anticipated that it would. In other words, this time, Lydia was able to hold the quantitative structure constant across different coordinate system orientations without having to rotate the graphs and carry out activity to establish invariance. So, we interpreted her activities of rotating of the graph and selecting specific points in her explanation to be mainly driven by her desire to explain her thinking, rather than a necessity to support her thinking. This stands in contrast with her activity in Task C (*Excerpt 2*), in which she engaged in sensorimotor activity to verify the invariance of the quantitative relationship between arc length and the vertical distance from the horizontal diameter of the circle for different orientations of the presented graph.

Looking across Lydia's activity in *Excerpts 3* and *4*, and in contrast to her activities in Task A, Lydia reasoned about the quantities and associated changes in each quantity by maintaining a commitment to directionality of measure comparison in accordance with the axes no matter the orientation of the paper and associated graphs. She did so both in terms of general variations in the quantities' values (*i.e.*, increase or decrease) and with respect to specific variations in the quantities' values (*e.g.*, *Excerpt 4*). Second, and relatedly, Lydia demonstrated an awareness of the invariant quantitative relationships (*e.g.*, she referred to one as a "positive, negative relationship" in *Excerpt 4*) associated with positive or negative "slope" so that these relationships were maintained across all orientations. Importantly, when reasoning about the "slope" of a line, Lydia did not physically re-orient the graph such that the y-axis was oriented in the canonical way (*i.e.*, vertically from her perspective), nor did she adhere to the visualization of "what we're used to a negative slope looking like," like she did in Task A.

From these features, we infer that Lydia's reasoning in Task D demonstrates a continued shift from maintaining figurative frames of reference to operative frames of reference. That is, Lydia's frame of reference in this task was not dominated by perceptual elements or (carrying out) sensorimotor experience associated with any particular coordinate system orientation. Instead, her claims of "slope" here were rooted in explicit attention to magnitudes organized in a directed, operative system of frames of reference she could coordinate across different orientations.

5. Discussion and implications

Through our analysis, we presented Lydia's establishing and maintaining frames of reference in the context of interpreting quantitative relationships in dynamic situations and graphing. Instead of focusing on what Lydia did or did not do successfully, we focused our analysis on modeling the processes through which she reasoned about quantitative relationships and in making distinctions between figurative and operative aspects of thought. Here we highlight our main findings, focus on particular shifts in Lydia's reasoning, and discuss broader implications of these in mathematics education.

5.1. Figurative frames of reference and operative frames of reference

Based on her activities throughout the tasks, we infer that Lydia engaged in establishing frames of reference and that her frames of reference were critical in her reasoning about quantitative relationships. In the aforementioned tasks, Lydia committed to a reference point which she used to gauge the relative position of quantities' values. These reference points were attributed to specific parts of the graph, such as the origin in Tasks A, C, and D or the intersection of the blue and red segment in Task B. Lydia also consistently committed to some directionality of measure comparison to determine change between quantities' values (*e.g.*, Lydia associated moving to the left horizontally with a negative change in a quantity's value in Task A). These frames of reference served as a mental structure she used to situate and reason about change or accumulation of quantities' magnitudes.

Although Lydia consistently established and used frames of reference when graphing, we made distinctions between her frames of reference based on the elements she established her frames of reference upon. In Tasks A and B, Lydia's commitment to a reference point or directionality of measure comparison relied heavily on sensorimotor elements (*e.g.*, moving to her left implies a decrease in values; above a segment implies positive magnitudes), in which case we referred to her frames of reference as figurative. While this way of reasoning would have enabled Lydia to work in what an observer might consider canonical coordinate system orientations, the actions enabled by reasoning within a figurative frame of reference are constraining in that they are not suitable for reasoning in non-canonical coordinate systems or reasoning about complex quantitative structures that rely on an operative understanding of coordinate systems and associated frames of reference. As evidence of this claim, these types of figurative commitments invoked perturbation for Lydia when reasoning about quantitative structures in non-canonical coordinate system orientations. For example, in Task A, Lydia conceived of inconsistencies in her point-wise coordination of quantities' measures and her "slope" coordination of the variation in quantities' measures. She was unable to reconcile these inconsistencies, which she expressed to be "so annoying."

On the other hand, in Task C and Task D, Lydia's commitment to a reference point or directionality of measure comparison gradually became divorced from sensorimotor elements. Lydia showed signs of establishing operative frames of reference, in which she committed to a stable reference point and directionality of measure comparison that were consistent with the non-canonical coordinate systems and rotations thereof. For example, in Task D, Lydia's frame of reference was not dominated by perceptual elements or carrying out sensorimotor activity associated with any particular coordinate system orientation. Instead, her claims of "slope" of the graph were rooted in explicit attention to magnitudes organized in a directed, operative system of coordinated frames of reference.

Returning to the spatial cognitionists' distinction between three different types of frames of reference discussed earlier, in Task A, Lydia's attendance to a figurative association with moving to *her* left implying decrease in quantities' measures is consistent with a viewer-centered perspective of the graphical representation. However, Lydia's later actions suggest her decentering from that perspective to gradually impute frames of reference to coordinate system representations that are not constrained to her immediate viewer-centered frame of reference. Moreover, in Task D, Lydia was able to reason about quantitative relationships in multiple coordinate system orientations without having to rotate the paper to align them with her immediate viewer-center perspective. Therefore, and consistent with Lee's (2017) findings from a study with four ninth-graders' constructions of coordinate systems in spatial contexts, we hypothesize that decentering and bringing forth images of one perspective alongside another acted as key cognitive resources in Lydia's shifts in her frames of reference. We contend that accounts of how students' actions relate to their conceived frames of reference are important because such accounts recognize that students are active mathematical thinkers. Moreover, such accounts provide insights into productive cognitive resources for students' mathematical learning that inform teaching, curriculum development, and research (Hackenberg, 2014), which we discuss next.

5.2. Shifts in Lydia's reasoning about quantitative relationships and mathematical tasks

Over the course of the teaching experiment, we observed shifts in Lydia's commitment to a reference point and directionality of measure in her quantitative reasoning. We contend that a potential explanation for this is that the environment in which Lydia engaged in novel graphing activity in non-canonical coordinate system orientations supported such shifts in her quantitative reasoning. Specifically, working in non-canonical coordinate system orientations invoked perturbations and afforded Lydia opportunities to operationalize a commitment to directionality independent of the perceptual features of coordinate systems and/or the graphical representations. Accordingly, we emphasize the importance of opportunities that afford students engaging in and differentiating between figurative and operative frames of reference. Focusing on a population of pre-service secondary mathematics teachers, and supporting Zazkis (2008) call to engage teachers in non-canonical situations, we believe our study highlights the importance of teacher education in which PSTs are given opportunities to challenge their ways of reasoning within quantitative frames of reference.

Coupled with the use of non-canonical coordinate systems, we also hypothesize that the questions from the TR to reconsider the graphs in various orientations and the PST's physical enactment of sweeping along or rotating the graphs supported reorganizations in her notion of reference point and associated directionality of measure comparison. For example, in Task B, the TR's pointing to the fixed point and prompting to repeat her reasoning seemed to have perturbed Lydia, illustrated by her comments, "I don't like what you're showing me right now" and, "Are you going to do the pen thing again and change my whole…", which provided an opportunity for re-organizations of her commitment to a reference point and directionality of measure comparison.

We acknowledge that Tasks A and D were different from Tasks B and C. Tasks A and D required committing to directionality of change in measure whereas Tasks B and C required committing to directionality of a measure comparison in accumulating quantities. Nonetheless, both types of tasks involved committing to directionality of some measure, and Lydia's experienced perturbations and demonstrated shifts were when she was attending to variation, either in each quantities' measures or in coordinate system orientations. Therefore, looking at a broader scope of distinguishing between figurative or operative commitments to directionality of measure comparison, we find both type of tasks to be worthwhile to consider. We also acknowledge that the tasks we used in the teaching experiment primarily involved pre-constructed graphs, pre-constructed representations, or pre-established formulas. These are different types of situations and future research might address these differences more intentionally, as well as investigating situations in which students are asked to construct graphs and formulas themselves.

5.3. Broader implications

Zooming out from our specific focus on Lydia's frames of reference in specific contexts, we contend that our findings have broader implications with respect to research and teaching related to students' quantitative and covariational reasoning. Although not an initial goal of the teaching experiment, we found shifts in Lydia's establishing and reasoning within frames of reference associated with the Cartesian coordinate system. We infer that as Lydia progressed through the teaching experiment, the Cartesian coordinate system became an increasingly operative structure in that she could coordinate numerous orientations of the Cartesian coordinate system and the multiple frames of reference that she imputed to them. Respective to such findings, we see a parallel in her progression in constructing abstracted quantitative structures. Recall that Moore and Silverman (2015) defined an abstracted quantitative structure as "a quantitative structure that becomes so internalized and operational that [a person is] able to assimilate novel representational contexts and situations to that structure" (p. 523). If we frame Lydia's activity in terms of her understanding of *slope* or *rate of change* (and *linear relationships*), her activity in Task A illustrates her first experiencing constraints in conceiving an invariant relationship—one she had likely experienced a canonical graph of numerous times throughout her educational experience—among various representations or situations that we designed to entail this relationship. As she progressed through the teaching experiment, her activity illustrated her becoming increasingly able to imbue particular (quantitative) operations to situations that differed perceptually from those she previously experienced; by adopting a focus on figurative and operative thought, we can interpret her activity in terms of her conceiving *slope* or *rate of change* as an operational structure that she could re-present and construct to assimilate novel situations.

Based on the above inferences and findings, we highlight the potential that using non-canonical coordinate systems and representations *in combination with* an emphasis on covarying quantities has in enabling researchers to make more detailed and nuanced claims regarding students' understandings than those claims when restricting their focus to canonical representations. Using such systems and representations enables researchers (and educators) to gain insights into students' abstractions (Piaget, 2001; von Glasersfeld, 1991) from their previous educational experiences including the extent those abstractions are constituted by operative thought. Furthermore, and turning our attention to educational implications of such use, doing so affords students opportunities to engage in and reflect upon a multitude of actions for the purpose of abstracting invariance across those actions. We echo researchers' calls (see Thompson, 1994; Johnson, 2015; Moore et al., 2013) to increase an emphasis on the role of such actions in the development of mathematical thought. Further studies investigating students' constructions of coordinate systems for reasoning about quantitative relationships with a specific attention to their frames of reference are needed.

5.4. Closing remarks

Returning to our opening discussion on frames of reference and work by spatial cognitionists, our findings illustrate the utility of adopting a line of inquiry that simultaneously makes explicit the perceiver of a situation and seeks to model the mental actions involved in the perceiver establishing frame or frames of reference. By characterizing differences in these mental actions, we provide insights into the process through which one reasons about quantitative relationships within frames of reference, how a student's frames of reference can entail figurative or operative commitments, and the potential implications of those commitments for their reasoning about mathematical ideas, such as function. Related to our foregrounding the active role of a perceiver in a way that makes the ontology of frames of reference salient, we believe the results of the study can provide insights for productive cognitive resources for mathematical activity involving the use of coordinate systems in quantitative reasoning. We envision that further investigations into students' conceptualization of frame(s) of reference with attention to figurative and operative thought will provide additional insights into the mental actions and reorganizations necessary to construct operative frames of reference. In turn, these insights can inform the design of instructional materials and experiences that afford and support students engaging in these mental actions and reorganizations, including for the purpose of constructing understandings of mathematical ideas (*e.g.*, function and graphing) that are productive for accommodating novel contexts.

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