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Within the line of work on students' quantitative reasoning, researchers have alluded to the significance of time in constructing covariational relationships. I draw on this body of literature and return to Piaget's perspective on time to provide a framework for the role of time in students' (co)variational relationships. The framework also clarifies the nature of the multiplicative objects underlying students' (co)variational relationships. In support of illustrating the framework and capturing its emergence from building second-order models of students' mathematics, I also describe a task and how its design reflects the framework.

Keywords: Covariational Reasoning, Time, Piaget, Quantitative Reasoning

Time has long been a topic of contemplation for researchers and philosophers, and ontological and epistemological considerations of time are certainly not restricted to the academy. Kant (1781/2003) considered time to be so ubiquitous as to be given a priori. Differing from Kant, Piaget viewed time as a concept an individual constructs. Accordingly, Piaget dedicated several studies to developing conceptual models of that construction (e.g., Piaget, 1954; Piaget, 1970). Based on his findings, Piaget proposed that the mental operations involved in constructing time are inseparable from space, motion, and objects (Piaget, 1970; von Glasersfeld, 1984). Building on researchers who have alluded to covariational reasoning being connected to time, I return to Piaget's (1970) conceptual models for time to further develop the role of time in students' (co)variational reasoning. In doing so, I elaborate on the constructs of *experiential time* and *conceptual time* (Castillo-Garsow, 2012; Thompson & Carlson, 2017) to provide a framework for characterizing students' (co)variational reasoning in relation to concepts of time. Reflecting its empirical roots, I illustrate the framework by describing a task designed to provide insights into the role of time with respect to students' (co)variational reasoning.

Covariational Reasoning and Time

The connection between motion, variation, and the concept of time has been indicated within work on students' covariational reasoning (e.g., Ellis et al., 2020; Johnson, 2015b; Paoletti & Moore, 2017; Patterson & McGraw, 2018; Stalvey & Vidakovic, 2015; Thompson & Carlson, 2017). Covariational reasoning—defined as the cognitive activities involved in reasoning about how quantities vary in tandem (Carlson et al., 2002; Saldanha & Thompson, 1998)—is an emergent area of research within the landscape of quantitative reasoning. Researchers exploring covariational reasoning have illustrated its importance for the learning of concepts spanning middle, secondary, and undergraduate mathematics (Byerley & Thompson, 2017; Carlson & Oehrtman, 2004; Ellis, 2011; Ellis et al., 2015; Johnson, 2015a, 2015b; Moore, 2014; Paoletti et al., 2023; Thompson et al., 2017), with other researchers identifying its broader importance in STEM (Gantt et al., 2023; Rodriguez et al., 2019; Sokolowski, 2020; Yoon et al., 2021).

With respect to relationships between time and covariation or function, researchers have primarily focused on time as a parameter (Keene, 2007; Kertil et al., 2019; Paoletti & Moore, 2017; Patterson & McGraw, 2018; Stalvey & Vidakovic, 2015; Trigueros, 2004). These researchers have focused on the extent to which time is held implicitly or explicitly in mind by students as they construct and reason about relationships between quantities. For instance, Patterson and McGraw (2018) explored student meanings in the context of dynamic situations and their graphing quantitative relationships that did not include elapsed time as a graphed quantity. Relatedly, Paoletti and Moore (2017) explored how graphing experiences with quantitative relationships not explicitly involving elapsed time can create an intellectual need for time as a parameter. Taking a different approach, Stalvey and Vidakovic (2015) focused explicitly on students constructing relationships between elapsed time and two other quantities, and then their subsequent construction of a relationship between those two quantities.

Some of the aforementioned studies drew on notions of conceptual and experiential time, which Castillo-Garsow (2012) and Thompson (2011, 2012) introduced to characterize students' (co)variation. Having roots in Piaget's (1970) framing of time and Newtonian mathematics (Thompson, 2012), conceptual and experiential time are akin but not identical to explicit and implicit parametric distinctions. Whereas parametric distinctions focus on time as a distinct quantity, conceptual and experiential time are organic to quantities' (co)variation. Rather than framing time as implicit or explicit attribute in and of itself, time is framed as an emergent, intrinsic property of (co)variation that differs based on the (co)variation conception. Thompson (2012) described experiential time as "felt time that [passes]" in an experience, while conceptual time is part of the "flowing" of quantities and "Not time on a clock, but an imagined, smoothly changing, quantified time—a measured duration that grows in extent" (p. 147). The distinction between experiential and conceptual time is situated in how a phenomenon's attributes are conceived, reflecting Piaget's (1970) distinction between intuitive time and operational time.

Linking Piaget's Cognitive Account of Time and Covariation

"We are far too readily tempted to speak of intuitive ideas of time, as if time, or for that matter space, could be perceived and conceived apart from the entities or the events that fill it" (Piaget, 1970, p. 1). Piaget considered time to be an emergent property of the co-ordination of simultaneous positions and the co-ordination of successive, spatial states. He referred to these co-ordinations as simultaneity and succession (with displacement), respectively, with their development occurring in the context of motions with different velocities. Piaget's view of time's link to conceptions of space and motion reflects his stance that concepts arise from the coordination and abstraction of mental actions. To Piaget, our temporal experience and memory of a situation are constructions subject to mental actions. We transition from intuitive to operative conceptions of time as we develop ways for organizing our experience that foreground operative forms of thought over experiential or figurative forms of thought (Piaget, 1970).

$$\begin{array}{ccccccccc} \mathbf{O_1} & \xrightarrow{a} & \mathbf{A_1} & \xrightarrow{a'} & \mathbf{B_1} & \xrightarrow{b'} & \mathbf{C_1} & \xrightarrow{c'} & \cdots \\ \downarrow o & & \downarrow o & & \downarrow o & & \downarrow o & & \\ \mathbf{O_2} & \xrightarrow{a} & \mathbf{A_2} & \xrightarrow{a'} & \mathbf{B_2} & \xrightarrow{b'} & \mathbf{C_2} & \xrightarrow{c'} & \cdots \\ \downarrow o & & \downarrow o & & \downarrow o & & \downarrow o & & \\ \mathbf{O_3} & \xrightarrow{a} & \mathbf{A_3} & \xrightarrow{a'} & \mathbf{B_3} & \xrightarrow{b'} & \mathbf{C_3} & \xrightarrow{c'} & \cdots \\ \vdots & & \vdots & & \vdots & & \vdots & & \end{array}$$

Figure 1. Piaget's co-seriation model of events, simultaneity, and succession. (Piaget, 1970, p. 264)

Piaget (1970) formalized the construction of simultaneity and succession of events as a grouping (i.e., co-seriation) shown Figure 1. $O_{\#}$ represents the initial state of event # (e.g., an attribute of an object/phenomenon like position in visual field, weight, or color). $A_{\#}$, $B_{\#}$, $C_{\#}$, and so on represent successive states of event #. a, a', b', c', and so on represent durations such that $b = a + a'$, $c = b + b'$, and so on. Piaget used $\downarrow o$ to link states of events occurring simultaneously (e.g., an object's weight and height), which can be thought of as a null vector due to the events' simultaneity. Piaget's (1970) model captures the multiplicative basis of co-seriation, in which events are united to form a *multiplicative object*—the cognitive uniting of attributes so that an object is simultaneously all of them (Inhelder & Piaget, 1964). As I illustrate below, constructing such an object is fundamental to the covariation of quantities (Saldanha & Thompson, 1998).

Drawing on Piaget's model of time and the simultaneity and succession of events, I present three conceptual models of time as it relates to an individual's conception of a phenomenon that entails quantities' magnitudes varying (e.g., $||x||$, $||y||$, $||z||$,…). The first model (Figure 2a) conveys a conception tied to experiential time. The second and third models (Figure 2b-c) each convey a conception tied to conceptual time. The second foregrounds the quantities as conceived with respect to elapsed time, while the third involves disembedding the quantities from the phenomenon and elapsed time so that they exist in an invariant relationship with each other.

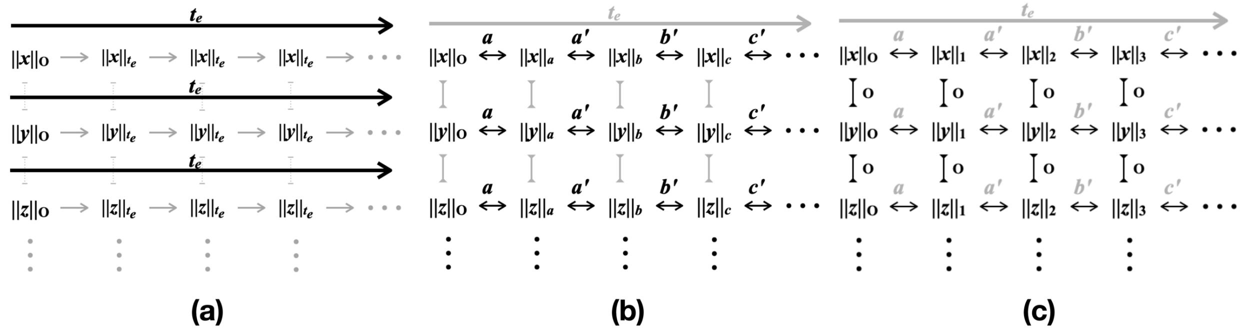


Figure 2. Conceiving a phenomenon and quantities (a) with respect to experiential time, (b) with respect to conceptual, elapsed time, and (c) so they are disembedded with respect to time and understood in terms of their invariant relationship.

Adopting expression notation and restricting the focus to two quantities, we can represent Figure 2a, Figure 2b, and Figure 2c with $||x||_{t_e} \vee ||y||_{t_e}$, $(||x||_t \vee ||y||_t)$, and $(||x||_\Delta \wedge ||y||_\Delta)$, respectively. I use $||x||_{t_e} \vee ||y||_{t_e}$ with $\vee$ (OR) and no parentheses to indicate that when a phenomenon and its constituent quantities are conceived with respect to experiential time, the quantities are both understood as present and varying in experience. They are observed to co-occur, but they are not cognitively linked beyond that. A conception of their relationship involves sequentially recalling and possibly, but not necessarily, comparing the intuitive, in-the-moment experience of each quantity's variation. This is captured by the weak link between $||x||$ and $||y||$ in Figure 2a and foregrounding experiential time, t_e, with each quantity's variation.

I use $(||x||_t \vee ||y||_t)$ and $(||x||_\Delta \wedge ||y||_\Delta)$ to indicate a phenomenon and its constituent quantities conceived with respect to conceptual time, whether elapsed (t) or their relationship disembedded and understood with respect to variation (Δ) from another state. With respect to $(||x||_t \vee ||y||_t)$, I use parentheses to indicate that the quantities are understood as occurring simultaneously, but I use $\vee$ to indicate that elapsed time is the driver of the relationship such that each quantity exists in a multiplicative object with elapsed time but not with each other. The two quantities are related through their sharing a relationship with elapsed time. This is captured by the link between $||x||$ and $||y||$ in Figure 2b, which is stronger than that in Figure 2a but mitigated by the connection to elapsed time. With respect to $(||x||_\Delta \wedge ||y||_\Delta)$, I use $\wedge$ (AND) and parentheses to indicate that the quantities are understood as occurring simultaneously and persistently. One quantity's magnitude is held in mind with the "immediate, explicit, and persistent realization that, at every [magnitude], the other quantity also has a [magnitude]" (Saldanha & Thompson, 1998, p. 298). The quantities' magnitudes are the driver of the relationship, and thus properties of the relationship are understood as defining the multiplicative object formed by joining the two quantities' magnitudes and thus are sustained irrespective of elapsed time or figurative aspects of experience. This is captured by the link between $||x||$ and $||y||$ in Figure 2c, which indicates their simultaneous and persistent co-existence so that their covariation is defined precisely by their simultaneous variations. Figure 2b and Figure 2c each indicates a bi-directional relationship between states to reflect the operational nature of conceptual time (Piaget, 1970). Figure 2c indicates measured durations fade to the background so the relationship is not tied to any particular experience or measured duration.

Illustrating the Framework - Time and Task Design

The task illustrated here emerged during a teaching experiment with undergraduate mathematics education students as part of a larger project focused on capturing middle grades and undergraduate students' reasoning within dynamic situations (see Liang and Moore (2021), Lee et al. (2019), Tasova and Moore (2020), and Moore et al. (2019)). With respect to the task below, the project team drew on two sources of inspiration beyond the second-order models of student thinking that emerged during the teaching experiment (Steffe & Thompson, 2000; Thompson, 2008). As one source, we drew on the tasks demonstrated by Saldanha and Thompson (1998) and Carlson et al. (2002) that involve covarying quantities other than time. Tasks that prompt students to construct graphs with respect to time make it difficult for a researcher to tease out whether the student is reasoning with respect to conceptual or experiential time (Thompson & Carlson, 2017). The task below includes two distances (i.e., magnitude bars that provide figurative material to enact quantitative and covariational operations) with no reference to elapsed time. Piaget's (1970) aforementioned work on time provided the second source of inspiration for the task. Piaget described, "It is only by the co-ordination of at least two motions with different velocities that purely temporal relationships can be distinguished from spatial relationships or from intuitive ideas about motion" (p. 26). The task foregrounds relations of simultaneity and succession via prompting the participants to coordinate two objects in motion, with the two objects varying at different rates with respect to elapsed time.

The Task: Which One? – Going Around Gainesville (GAG)

"Which One? – GAG" is from a series of tasks titled "Which One?" A "Which One?" task is designed to be implemented after a participant constructs a covariational relationship within phenomenon or a graphical representation (Liang & Moore, 2021). A "Which One?" task provides several representations of covariational relationships, including magnitude bar sets that vary simultaneously or a collection of static or dynamic graphs. With the representations provided, the researcher asks the participant which of the representations, from none to all, accurately capture the relationship they identified previously (whence the name, "Which One?").

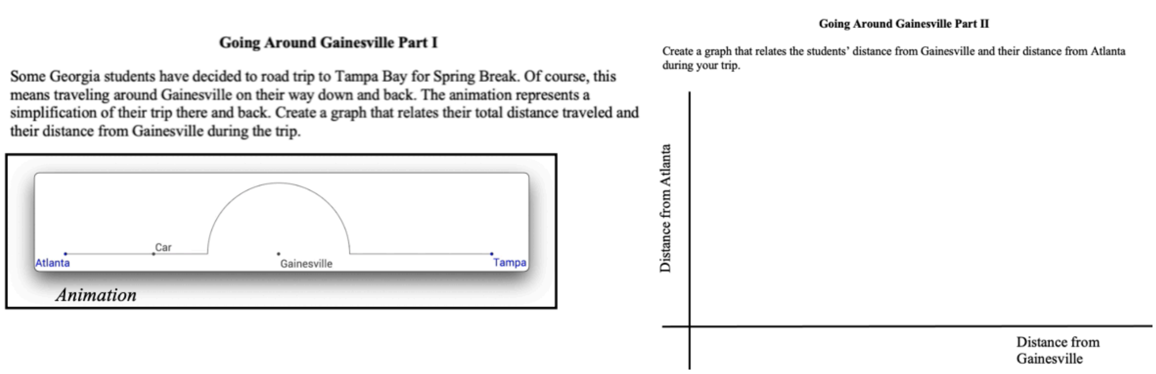
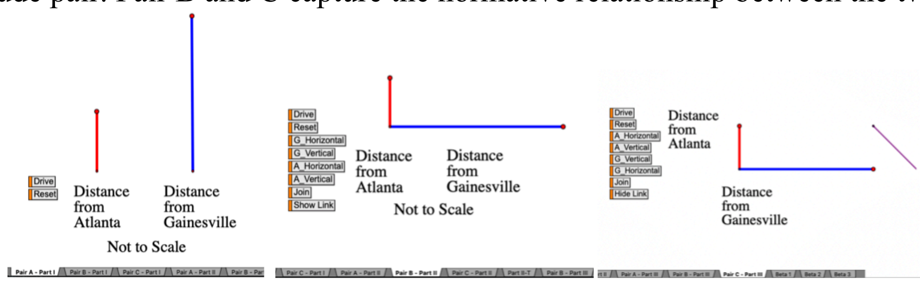


Figure 3. The Going Around Gainesville (GAG) task, video at: https://youtu.be/v2yc55Z9WV8.

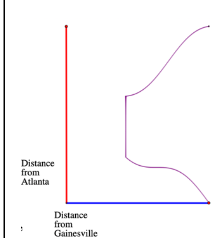
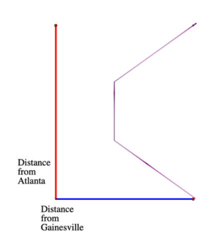
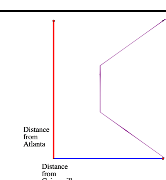
The part preceding "Which One? – GAG" involves a video depicting a car starting in Atlanta and traveling back and forth from Tampa (Figure 3, see Moore et al. (2022) and Moore et al. (2019) for empirical data). After viewing the animation, the participant is sequentially asked two graphing tasks (Figure 3). After a participant engages in each part and has constructed what the research team perceives to be a stable understanding of the covariational relationship, the researcher implements the three-part task "Which One? – GAG". Each part consists of three pairs of magnitude bars presented in a *dynamic geometry environment* (DGE). As support for the reader, https://tinyurl.com/4v9ma7pc hosts videos illustrating each part and pair of the task. For Part I of the task (see Figure 4a for a snapshot), the participant is presented with three tabs, each containing a pair of magnitude bars. For each pair, one magnitude bar represents the *distance from Atlanta* (*dfA*) and one magnitude bar represents the *distance from Gainesville* (*dfG*). For each pair, the student can push "Drive" to start or stop the bars changing together, and the student can push "Reset" to return the pair to a zero-magnitude *dfA* and corresponding initial *dfG*. The participant is tasked with determining which, if any, of the pairs covary as to accurately capture the determined relationship between the *dfA* and the *dfG*. Table 1 describes the design of each magnitude pair. Pair B and C capture the normative relationship between the two distances.



(a) (b) (c)

Figure 4. Example still shots for (a) Pair A – Part I, (b) Pair B – Part II, and (c) Pair C – Part III.

Table 1. The design of "Which One? – GAG".

RELATIONSHIP DESIGN	PART III
Pair A: With respect to *dfA*: *dfG* decreases at an increasing rate, decreases at a decreasing rate, remains constant, increases at a decreasing rate, and then increases at an increasing rate. When Drive is pushed, with respect to elapsed time: (i) *dfA* increases at a decreasing rate, increases at an increasing rate, increases at a decreasing rate, increases at an increasing rate, and then increases at a decreasing rate. (ii) *dfG* decreases at a constant rate, remains constant, and then increases at a constant rate.	
Pair B: With respect to *dfA*: *dfG* decreases at a constant rate, remains constant, and increases at a constant rate. When Drive is pushed, with respect to elapsed time: (i) *dfA* increases at a decreasing rate, increases at an increasing rate, increases at a decreasing rate, increases at an increasing rate, and then increases at a decreasing rate. (ii) *dfG* decreases at a decreasing rate, decreases at an increasing rate, remains constant, increases at an increasing rate, and then increases at a decreasing rate.	
Pair C: With respect to *dfA*: *dfG* decreases at a constant rate, remains constant, and increases at a constant rate. When Drive is pushed, with respect to elapsed time: (i) *dfA* increases at a constant rate. (ii) *dfG* decreases at a constant rate, remains constant, and then increases at a constant rate.	

Part II of the task (see Figure 4b for a snapshot) presents the same three pairs of magnitude bars, but they can reorient the magnitude bars, join then, and show a "link" between them. This link represents the process of joining two orthogonal magnitudes to form a Cartesian point. A participant is told that each pair matches its respective pair from Part I (e.g., Pair A in Part I, II, and III covary equivalently), and that Part II of the dynamic sketch is designed to help them further explore the extent the two magnitude bars capture the determined relationship between the two distances. For Part III of the task (see Figure 4c for a snapshot), the participant is again presented with the same three pairs of magnitude bars. In this case, each pair is oriented orthogonally, a Cartesian point is displayed, and a trace of the point is recorded as the magnitude bars covary. Like Part II, the participants are told that each pair matches its respective pair from Part I, and that Part III is to aid further exploring the extent the two magnitude bars capture the appropriate relationship between the two distances. During Part II and Part III, a participant is also prompted to reflect on and describe any changes in their assessment of the paired magnitudes. They can return to the previous parts if desired. They are also asked to reflect on difficulties from previous parts and how subsequent parts assist their assessment. Said frankly, Part I is intended to be difficult, both conceptually and in functional design, with the hopes of both eliciting their thinking and affording spontaneous requests for other representations.

Connecting the Task to the Framework

First focusing on Figure 2a (i.e., $||x||_{t_e} \vee ||y||_{t_e}$), and reflecting quantities' variations occurring in experiential time, a student reasoning in such a way attends to the variation of each magnitude separately, and they primarily do so through the experience of watching the DGE animated continuously using "Drive". With respect to Pair A, the student might conclude that *dfG* varies appropriately due to its smooth decrease, constancy, and then increase, while concluding that *dfA* varies incorrectly. For the latter, they anticipate that *dfA* increase at a smooth rate, which reflects the manner in which it increases during the experience of watching the road trip animation. With respect to Pair B, and consistent with their response to Pair A, the student might conclude that *dfG* and *dfA* vary inappropriately due to anticipating both increases or decreases at smooth rates, again reflecting how they experience the variations with the road trip animation. With respect to Pair C, the student is likely to conclude that both *dfG* and *dfA* vary appropriately due to the smooth variation of each. Across all of the pairs, the student primarily focuses on each magnitude separately and draws on intuitive or experiential notions of rate to draw conclusions.

For Figure 2b (i.e., $(||x||_t \vee ||y||_t)$), due to the basis in conceptual time, a student reasoning in such a way attends to the variation of each magnitude separately, but they coordinate the variation of each using successive durations of elapsed time. This might be accomplished by stepping through states of the DGE and tracking the variation of each quantity with anticipated properties in mind. With respect to Pair A, as the student tracks through successive, equal duration states of the DGE, the student might conclude that although *dfG* varies by constant amounts, *dfA* does not vary by constant amounts and thus the magnitude bars do not capture the appropriate relationship. With respect to Pair B, the student might comment on the difficulty assessing the pair using the DGE and thus seek to step through the DGE state by state. Reflecting that the quantities are cognitively linked through their shared relationship with elapsed time in this form of covariation, the student might attempt to "Drive" the bars for equal durations of time and then compare the variations of the magnitudes to each other. With respect to Pair C, the student is likely to conclude that the pair covaries appropriately due to the smooth variation of each, and they might further test this by using successive, equal durations of "Drive". Across all of the pairs, the student coordinates each magnitude with equal durations in order to draw comparisons across the magnitudes. Because of this, Pair B can lead to a perturbation that stems from the student anticipating equal variations in each quantity for equal variations in duration due to the piecewise linear relationship between *dfG* and *dfA*.

For Figure 2c (i.e., $(||x||_\Delta \wedge ||y||_\Delta)$), due to the basis in a disembedded invariant relationship, a student reasoning in such a way foregrounds coordinating a quantity's variation with respect to the other quantity's variation. Whether Pair A, B, or C, the student is likely to attempt to vary one quantity's magnitude in a systematic way while tracking the variations in the other quantity's magnitude. For instance, the student might use "Drive" to step *dfA* through successive, equal increases, and then assess the appropriateness of the pair by investigating whether the *dfG* magnitude follows the pattern of constant decrease, constant, and constant increase. A student engaging in such covariational reasoning might experience a perturbation stemming from the functionality of the DGE (e.g., it is difficult to use "Drive" to step through equal amounts of *dfA* increase), but they would not be significantly perturbed by how a single bar varies as the animation plays. They persistently foreground how the bars simultaneously covary, which can lead to expressing annoyance at Part I and motivating a need for Parts II-III and a graph.

Closing

The three forms of (co)variational reasoning differentiate (co)variation based on the role of time and, hence, the extent a multiplicative object is formed between the two quantities. The three forms invite questions regarding their developmental and hierarchical nature. The three forms emerged from work conducted primarily with undergraduate students, and I do not have second-order models of their developmental trajectory and relationships. I hypothesize the continued work by colleagues such as Ellis, Johnson, Lee, Paoletti, and Tasova will provide such insights. With respect to hierarchy, there is a relative increase in sophistication and generativity from Figure 2a to Figure 2c that is reflected in Piaget's exposition of time, as well as Carlson, Castillo-Garsow, Saldanha, and Thompson's descriptions of (co)variation. This relativeness is captured by Patterson and McGraw (2018), who described,

> We hypothesize that it is advantageous to be able to envision the covariation between two dynamically changing quantities and, to some degree, decouple this image of covariation from a unidirectional, experiential image of the passage of time. This process is essential for developing an understanding of an invariant relationship between two quantities and explaining how changes in one variable constrain changes in another variable. (p. 320)

The authors hedge in their hypothesis, as the process of decoupling quantities' covariation from experiential time is intrinsic to the form of covariation captured in Figure 2c and, more broadly, that suggested by Carlson, Castillo-Garsow, Saldanha, and Thompson. Constructing a multiplicative object between quantities' magnitudes necessarily involves decoupling images of variation from experiential or specific passages of elapsed time. It is then that two quantities' variations are taken as objects of thought and united so that an invariant relationship is constructed to constrain the two quantities' simultaneous variations. Although the forms have a hierarchical nature, the implications of such remain an open question. This is particularly true as it relates to how the forms of (co)variation play a role in students constructing concepts in which covariational reasoning provides a foundation, such as rate of change and accumulation.

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