

Chapter 4

Operationalizing Figurative and Operative Framings of Thought



Kevin C. Moore, Irma E. Stevens, Halil I. Tasova, and Biyao Liang

Introduction

Theory is in the end, as has been well said, the most practical of all things, because this widening of the range of attention beyond nearby purpose and desire eventually results in the creation of wider and farther-reaching purposes and enables us to use a much wider and deeper range of conditions and means than were expressed in the observation of primitive practical purposes. For the time being, however, the formation of theories demands a resolute turning aside from the needs of practical operations previously performed. (Dewey, 1929)

Theory is the stuff by which we act with anticipation of our actions' outcomes and it is the stuff by which we formulate problems and plan solutions to them. (Thompson, 1994b, p. 229, in reference to Dewey's quote)

K. C. Moore (✉)

Department of Mathematics, Science, and Social Studies Education, University of Georgia, Athens, GA, USA

e-mail: kvcmoore@uga.edu

I. E. Stevens

Department of Mathematics and Applied Mathematical Sciences, University of Rhode Island, Kingston, RI, USA

e-mail: irma.stevens@uri.edu

H. I. Tasova

Department of Teacher Education and Foundations, California State University, San Bernardino, San Bernardino, CA, USA

e-mail: halil.tasova@csusb.edu

B. Liang

Academic Unit of Teacher Education and Learning Leadership, The University of Hong Kong, Hong Kong, SAR, China

e-mail: biyao@hku.hk

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One framing of theory is that it provides ready-made tools to be applied as ways to understand, organize, and explain phenomena. In such a framing, theoretical constructs are mostly static in their definitions, with researchers using them as defined. For instance, Moore (2014a) used Carlson et al.' (2002) covariation framework to model a student's construction of the sine relationship in graphical and unit circle contexts. In adopting the mental actions introduced by Carlson et al. (2002), Moore did not make modifications to their definitions, instead applying them as defined in order to provide a hypothetical account of the cognitive actions driving the student's activity.

Another framing of theory is that it provides malleable tools for inquiry and research pursuits. Rather than providing ready-made tools to be applied, theory and theoretical constructs provide a general roadmap or orientation. Theoretical constructs are dynamic in their definitions, with researchers adapting constructs as they carry out empirical work so that the researcher can operationalize theoretical constructs in the context of novel settings and data. The adaptations result in an evolution of the theoretical constructs that account for novel settings while maintaining some central tenets of those constructs. For example, in adopting Ellis's generalization framework (Ellis, 2007b), Ellis and colleagues (2022) found it necessary to adapt that framework to model the reasoning of students across different grade bands and content areas. We write this chapter in the spirit of this latter framing.

The notions of figurative and operative thought were introduced over a half-century ago by Piaget (1969; Piaget & Inhelder, 1971, 1973), and since then, mathematics education researchers have used, extended, and shaped them in several ways. Their uses have been varied, and no singular definition of either sufficiently captures its development and use. Thus, in this chapter, we present various uses of the two notions of thought with the intention of highlighting their value in modeling student thinking. In doing so, we situate evolutions in the two constructs, while underscoring common tenets that span across these evolutions. We discuss methodological concerns regarding notions of figurative and operative thought, including how the constructs can be used in task design of empirical studies. We also discuss the constructs' implications for researchers' claims regarding students' meanings. Because theoretical constructs should be continuously tested in new areas to assess their viability and expand their generalizability, we end the chapter with various ideas for future research that can contribute to continued evolutions in the notions of figurative and operative thought.

Some “Definitions”

As mentioned in the introduction, we do not intend to provide definitive definitions of figurative and operative thought. This choice is not meant to imply that definitions are not important. Definitions help paint a picture of the constructs' central tenets when combined with researchers' uses of the constructs, but what ultimately

matters is the use of the constructs. As a resource to the reader, we provide a non-exhaustive collection of researcher quotes related to defining figurative and operative thought (Table 4.1).

Uses and Evolution of Figurative and Operative Thought

As can be inferred from Table 4.1, one of the more difficult aspects of reading Piaget is that he did not always provide concise definitions, and when he did, his definitions evolved to capture changes in his conception of knowledge and knowing. Piaget's approach to his writing and work mirrors his genetic epistemology and his conception of objects. To know an object is to act on it or engage in its use, as an object is comprised of operations of thought. Or, "To my way of thinking, knowing an object does not mean copying it—it means acting upon it" (Piaget, 1970, p. 15). It is through acting on an object (e.g., operationalizing a construct) that an individual furthers their understanding of the object via the mental operations imbued to the object.

Fortunate to our role as authors of this chapter, the figurative and operative distinction did not see significant evolutions in Piaget's writing and use. We hypothesize that this is because the distinction formed a general categorization of different knowledge forms, as opposed to constructs that emerged through nuanced characterizations of knowledge development (e.g., his various forms of abstraction). In fact, references to figurative or operative forms of thought occur in the context of Piaget describing forms of abstraction, in which he situates operative forms of thought with more sophisticated forms of abstraction.

As Montangero and Maurice-Naville (1997) reported, the figurative and operative distinction emerged during writings by Piaget and his colleagues on perception, representation, mental imagery, and memory (Piaget, 1969; Piaget & Inhelder, 1971, 1973). Defined generally, figurative aspects of thought include those things that involve perception, sensation, sensory objects, and physical motion. Figurative aspects of thought deal with configurations and states, of which an individual looks to mimic in production. Operative aspects of thought include the coordination of mental actions and their transformations, and are those mental operations that define objects and allow individuals "to free themselves from the illusions and deformations aroused by perception" (Montangero & Maurice-Naville, 1997, p. 144). Operative forms of thought enable a student to isolate numerosity or make compensations of dimensions when engaged in Piaget's famous pebble counting or glass and volume tasks. Figurative forms of thought lead to children relying on perceptual properties (including spatial positioning of endpoints) to draw conclusions about numerosity or volume.

Before transitioning to the ways in which the figurative and operative distinction has been used in depth by mathematics educators, we perceive Piaget's distinction to be related to (and have likely informed) his work on children's conception of

Table 4.1 Figurative and operative quotes

Figurative and operative thought quotes	
Figurative thought quotes	Source
Piaget did admit an inferior species of knowledge that merely pictured static configurations; he called the lesser kind <i>figurative</i> knowledge. For Piaget perception and—usually—language are figurative.	Piaget (2001, p. 2)
Figurative knowledge is knowledge of observables...	Piaget (2001, p. 298)
‘Figurative’ refers to the domain of sensation and includes sensations generated by motion (kinaesthesia), by the metabolism of the organism (proprioception), and the composition of specific sensory data in perception.	von Glasersfeld (1995, p. 69)
[the figurative aspect of thought] tends to include the figural character of reality, i.e., configurations as such. With it can be grouped: (a) perception, which functions only when an object is present and through the intermediary of a sensorial field; (b) imitation in the broad sense of the term (gestural, phonic, graphic, etc.) which functions either with or without the presence of the object, but in any case through overt or covert motor reproduction; (c) the mental image, which functions only when there is no object present.	Piaget and Inhelder (1971, pp. 11–12)
The figurative functions have no tendency to transform objects, but tend to supply imitations of them...static configurations, which are relatively easy to translate into images; even when they concern movements or transformations, they do so in order to produce the appropriate configurations, not the changes of state.	Piaget and Inhelder (1973, p. 9)
When a person’s actions of thought remain predominantly within schemata associated with a given level (of control), his or her thinking can be said to be figurative in relation to that level.	Thompson (1985, p. 195)
Adopting Thompson’s framing, a researcher drawing distinctions between figurative and operative thought is thus an issue of characterizing whether an individual’s meanings are tied to carrying out particular actions and their results...	Moore et al. (2019b, p. 2)
Operative thought quotes	
For Piaget, knowledge is not a matter of pictures or sentences or symbolic data structures... knowledge is fundamentally <i>operative</i> ; it is knowledge of what to do with something under certain possible conditions. Or it is knowledge of what that thing will do under different conditions... operative knowledge consists of <i>cognitive structures</i> .	Piaget (2001, pp. 2–3)
...whereas operative knowledge, which pertains to actions and operations, is knowledge of transformations. Now, logico-arithmetic operations cannot be rendered figuratively, unless this is done symbolically.	Piaget (2001, pp. 298–299)
The operative aspect of thought deals not with states but with transformations from one state to another.	Piaget (1970, p. 14)
In contrast, any result of conceptual construction that does not depend on specific sensory material but is determined by what the subject does, is ‘operative’ in Piaget’s terminology.	von Glasersfeld (1995, p. 69)

(continued)

Table 4.1 (continued)

Figurative and operative thought quotes	
Figurative thought quotes	Source
The operative aspect on the other hand takes in those forms of cognitive experience or of deduction whose function consists of modifying the object in such a way as to apprehend transformations as such. This includes: (a) sensorimotor actions (with the exception of imitation), the sole instruments of sensorimotor intelligence to be organized before language; (b) internalized actions that prolong previous ones right from a preoperational level; (c) the operations proper of the representational intelligence, or reversible internalized actions which organize themselves as a set of structures or as transformation systems.	Piaget and Inhelder (1971, pp. 11–12)
When the actions of thought move to the level of controlling schemata, then the thinking can be said to be operative in relation to the level of the figurative schemata.	Thompson (1985, p. 195)
...[operative thought] foregrounds the coordination of internalized mental actions so that figurative aspects of thought are subordinate to this coordination... the individual can call forth and control a scheme and its results.	Moore et al. (2019b, p. 2)

geometry (Piaget et al., 1981). Recall that to Piaget, objects are defined by operations of thought and thus the understanding of geometric objects involves moving beyond the observable. Despite the ease at which the study of geometric objects can be framed as the study of the observable, Piaget’s work underscores that the construction of geometric objects involves dissociating operative forms of thought from their figurative content so that the operative takes precedence over its figurative content, forms, and states (Norton, 2022; Piaget et al., 1981). Take, for instance, a circle. While the reader can likely evoke an image of a circle with ease, the mental coordination of rotating a fixed length about a fixed point and anticipating the trace of the rotating end-point defines the circle. We pull this tenet—one that involves identifying whether it is figurative or operative aspects of thought that take precedence in one’s meanings—through the rest of the chapter.

Transitioning the Constructs to Mathematics Education

Piaget and his colleagues’ work has influenced many mathematics educators’ research programs. Whether it be aspects of his genetic epistemology, his discussion of schemes and operations, his identified stages of development, or his framing of abstraction, Piaget’s body of work has proved fruitful in providing mathematics educators with tools to research, model, and promote individuals’ mathematical development. With respect to the figurative and operative distinctions, von Glasersfeld (1987, 1995) and Steffe (1991a, b; Steffe & Olive, 2010), as close collaborators, were early adopters who did not make significant adaptations to the distinctions.

von Glasersfeld (1995) incorporated Piaget's distinction in his exposition on the nature of knowing and learning in order to distinguish between two forms of thought with categorical differences. Consistent with Piaget, he explained that figurative thought involves sensation, perception, sensory objects, and physical motion, and is thus directly related to any abstractions based in specific sensory material, including motor signals. He contrasted this by explaining that operative thought involves conceptual operations and their coordination. He explained that the enactment of such operations occurs in the context of specific figurative material, their abstraction is such that they are not dependent on specific sensory material, and these abstractions can provide the figurative material for subsequent abstractions. At a more specified level, one of von Glasersfeld's (1987, 1995) primary interests was with representation, whether in terms of language, icons, or symbols. In reference to the figurative and operative distinction, von Glasersfeld (1987) explained,

With regard to icons, Piaget's distinction between the "figurative" and the "operative" would seem to be of some importance. Number is not a perceptual but a conceptual construct; thus, it is operative and not figurative. Yet, perceptual arrangements can be used to "represent" a number figuratively. Three scratches on a prehistoric figurine, for instance, can be interpreted as a record of three events. In that sense, they may be said to be "iconic"—but their iconicity is indirect. They do not depict "threeness", they merely provide the beholder with an occasion to carry out the conceptual operations that constitute threeness (Glasersfeld, 1981, 1982). Carrying out these operations does not involve *reference* to some prior sensorimotor item or elements of such items—it is the operating itself that each time constitutes the abstract conception of threeness...

An analogous distinction must be made in the case of symbols. On the one hand, there are symbols that refer to figurative items or sensorimotor situations, such as the King of France or the act of smoking; on the other, there are symbols that do not refer to sensorimotor experience at all but are merely indicators that a certain conceptual operation is to be performed. I would call this second category operative symbols and would list among them not only number words, numerals, and mathematical signs, such as "+", "—", and "=", but also prepositions, conjunctions, and certain other words whose interpretation does not depend on the recall of sensorimotor experiences but requires the construction of some operative conceptual relation. (Glasersfeld, 1987, pp. 7–8)

von Glasersfeld (1987) described that an icon refers to another item or experience via sensorimotor similarity, whereas a symbol refers to other items or experiences more arbitrarily. With respect to both icons and symbols, von Glasersfeld drew on Piaget's figurative and operative distinction to categorize those icons and symbols that form or refer to figurative items and sensorimotor situations, and those icons and symbols that (possibly in addition to forming figurative items) occasion or point to the coordination of mental operations.¹ In von Glasersfeld (1995), he echoed this distinction in reference to words. He explained that an individual's meaning for a word can be abstracted from sensorimotor experience and thus be figurative, or it can point to a coordination of mental operations and thus be operative. In the context of words, he added that a symbol or icon might merely cause a response in the

¹It is important to note that von Glasersfeld used symbol differently than Piaget as he described in his writings on radical constructivism (von Glasersfeld, 1995).

form of a carried-out action, a case in which the symbol or icon acts as a signal as opposed to an abstracted coordination of mental operations (Moore, 2014b; Glasersfeld, 1995). von Glasersfeld's application of Piaget's distinction to icons, symbols, and words is reflected in graphical associations related to static and emergent shape thinking discussed in section "[Informing generalized models of student thinking](#)".

With respect to Steffe and his colleagues, they adopted the figurative and operative distinctions to characterize children's development of schemes in the context of counting and quantities, including length. With respect to the example of length, Steffe and Olive (2010) identified that figurative length entails a visualized path, motion producing the path, and some sense of the duration of the motion. Steffe and Olive (2010) further identify that figurative length provides the foundation for an individual to construct quantitative properties through the individual's actions and reflection upon those actions in order to construct a system of coordinated operations (i.e., an operative meaning). Steffe and Olive's characterization of length underscores that characterizing a meaning as figurative need not carry negative connotations. Instead, a figurative meaning may be a natural part of conceptual development as an individual transitions from meanings based in perception, sensation, and motor actions to meanings based in mental actions introduced and eventually coordinated by the individual.

With respect to counting, and situated within children's construction of number, Steffe and colleagues (1991b; Steffe & Olive, 2010) distinguished between figurative and operative counting schemes. Whereas figurative length primarily foregrounded notions of motion and duration, a figurative counting scheme requires re-presenting perceptual material or engaging in some form of sensorimotor action when counting.² In order to support their counting of objects, especially if hidden, a child with a figurative counting scheme will produce or imagine material that is visual, such as an array of objects, or that is sensory, such as pointing in space or tapping their finger (Steffe, 1991b). Steffe and Olive (2010) introduced the term *figural unit items* for these re-presentational objects due to their basis in figurative material and sensorimotor action. An operative counting scheme is one in which the individual has constructed "a sequence of abstract unit items that contain the records of the sensory-motor material" (Steffe, 1991a, p. 35). Such a construction occurs through the individual coordinating the mental operations involved in counting so that they no longer need to produce or imagine figurative material or sensorimotor actions. The individual thus constructs unitized records of counting that they can then operate on to construct more sophisticated notions of quantity. Important to the figurative and operative distinction, including Thompson's extension described below, these unitized records or abstract unit items are not tied to specific

²We use "re-present" (or "re-presentation") and "representation" in distinct ways. Drawing on Piaget and von Glasersfeld (von Glasersfeld, 1995), as well as Liang and Moore (2021), we use "re-presentation" to refer to the enactment and regeneration of schemes and operations. We use "representation" in the canonical sense to refer to the modes of display and symbolization associated with the field of mathematics (e.g., graphs, inscriptions, and verbal statements).

re-presentational actions and associated figurative material. The regeneration of figurative material is necessary to re-present the records of counting, but it is the abstract unit items and unitized records of counting that drive the regeneration of that material.

Transitioning the Constructs to Higher Level Mathematics

For Steffe and colleagues, the figurative and operative distinctions proved fruitful in describing number meanings and the extent figurative material or sensorimotor activity were a necessary aspect of counting actions. Their distinction provided a productive avenue to expanding Piaget's discussion of concrete-operational intelligence (e.g., thought involving logical operations but limited to concrete or physical situations), which was sensible given that the development of number and counting schemes was their content area of study. This distinction is not as useful when discussing students' meanings for more advanced mathematics topics built on complex, intertwined systems of meanings, representational activity, and symbolization. Researchers have thus reframed the figurative and operative thought distinction as illustrated by the work of Thompson, Moore, and colleagues (Liang & Moore, 2021; Moore, 2021; Moore et al., 2019a, b; Thompson, 1985).

Thompson (1985) described Piaget's distinction as one of "the most significant that I know of for mathematics education" (p. 194). He proposed generalizing notions of figurative and operative thought to any level of meanings in order to account for the developmental nature of mathematical meanings, and the fact that a system of meanings at one developmental level can become the source material (i.e., the figurative ground) for a system of meanings at a subsequent developmental level. Thompson explained,

When a person's actions of thought remain predominantly within schemata associated with a given level (of control), his or her thinking can be said to be figurative in relation to that level. When the actions of thought move to the level of controlling schemata, then the thinking can be said to be operative in relation to the level of the figurative schemata. That is to say, the relationship between figurative and operative thought is one of figure to ground. Any set of schemata can be characterized as figurative or operative, depending upon whether one is portraying it as background for its controlling schemata or as foreground for the schemata that it controls. For instance, the thinking of a college mathematics major in an advanced calculus course, which certainly would be classified as being formal operational in Piaget's fixed sequence of cognitive development, could nevertheless be classified as figurative regarding the kind of thinking required in a graduate course in real analysis. Of course, we would have to make apparent to ourselves the possibility that the "objects" of such a student's thinking are things like functions, classes of functions, and associated operations. (Thompson, 1985, p. 195)³

³Here, we believe Thompson's use of schemata is consistent with scheme or meanings than his use of schema (see Thompson, 2013b). For a more extensive discussion of schemes, see Tillema and Gatzka (Chap. 3).

Thompson's discussion of operative and figurative thought reflects his interest in characterizing complex systems of knowing that entail a web of meanings that build on themselves during a student's development. Importantly, this development can occur both in the short-term and over a long period. For example, in the short-term, a student could construct a directional relationship (see Carlson et al., 2002) between two covarying quantities, which then can become the material on which they operate to construct an amounts of change relationship. Alternatively, over a longer period, a student could construct particular additive partitioning schemes that then become the source material for making multiplicative comparisons and constructing rate of change meanings (Byerley, 2019; Thompson, 1994a). Importantly, by identifying that what is operative at one level can become figurative at the next level as the individual attempts to operate on prior abstracted mental actions, Thompson provides researchers a way to characterize students' actions or meanings as figurative or operative no matter the mathematical content or "object."

Also expanding the applicability of figurative and operative thought, Thompson's reframing breaks free from a distinction resting on the availability or production of figurative material and sensorimotor actions. Thompson instead foregrounds the extent an individual's meanings are tied to carrying out particular actions and obtaining particular results, as opposed to their being able to transform their actions to account for novel experiences and associated figurative material. Said another way, Thompson's framing differentiates the focus of reasoning being properties of figurative material itself (including the results of activity) from the focus of reasoning being the mental actions and their coordination that *generate* associated figurative material and results.⁴

Thompson broadened the figurative and operative distinction to allow for the presence of figurative material (a point upon which we subsequently expand upon as it relates to task design), but it is not incompatible with Steffe and Olive's description of figurative counting schemes. A child with a figurative counting scheme is constrained to carrying out particular actions, and specifically that of producing sensorimotor actions or using available perceptual material. On the other hand, a child having constructed a sequence of abstract unit items is in a position to use those items in the presence of novel experiences and associated figurative material.

Thompson's reframing is also faithful to tenets of Piaget's use of figurative and operative thought. Piaget included imitation in his characterization of figurative aspects of thought but with a focus on sensorimotor reproduction. Piaget also noted that figurative aspects of thought involve imitations of objects and the production of configurations and observables. Thompson extends Piaget's notions of imitation and reproduction to include those meanings or actions that foreground imitating or habitually reproducing previously enacted actions, whether mental or sensorimotor, for the purpose of obtaining particular results and states. As Ellis et al. (Chap. 6) explain, such meanings are often tied to abstractions based on the results or products

⁴We note that this framing from Thompson is related to various forms of abstraction. For more information on those forms, see chapters by Ellis et al. (Chap. 6) and Tallman (Chap. 8).

of mental actions, as opposed to the coordination of the mental actions that produced particular results and objects (Moore, 2014b; Moore et al., 2019a, b). With respect to operative thought, Piaget's descriptions foregrounded internalized actions, organized and logico-mathematical structures, and an individual's ability to reason about transformations. This is consistent with Thompson's use of operative, with Thompson adding that what is operative on one level can become figurative in nature during subsequent experiences and development. Importantly, Thompson's addition is consistent with the goal-directed foundation of Piaget's genetic epistemology (Piaget, 1970; Steffe, 1991a; Glasersfeld, 1995).

Thompson's reframing has enabled researchers to characterize marked differences in students' meanings for the representations and symbolizations they construct, including the extent those meanings are tied to carrying out particular actions and their results. To illustrate the usefulness of Thompson's framing of figurative and operative thought in this regard, we first draw on results from investigations of students' graphical meanings and activity. This line of research has been prevalent because graphing necessarily entails aspects of both figurative and operative thought (Moore et al., 2019b).

Before we report on data and empirical studies that draw on Thompson's framing, we provide a few concise definitions and resources (Table 4.2) that will be of use to the reader. We include a list of relevant citations with each definition for the reader interested in reading more detailed works that use the associated term or construct. The connection between notions of figurative thought, operative thought, quantitative reasoning, and covariational reasoning is not by chance. Quantitative and covariational reasoning are apropos examples of operative thought, and specifically the enactment of logico-mathematical operations and associated meanings (Moore, 2019b; Norton, 2014; Thompson, 1985). We attempt to provide examples in the following sections so that they do not require a significant background in quantitative and covariational reasoning, but familiarity with research in those areas will undoubtedly assist the reader in situating our illustrations of figurative and operative notions of thought.

Models of Students' Graphical Thinking

Moore, Stevens, et al. (2019b) adopted Thompson's (1985) reframing of figurative and operative thought to characterize prospective secondary teachers' (PSTs') graphical meanings. The authors did so in order to highlight a marked difference in the PSTs' actions as they engaged in solving particular problems involving the interpretation or construction of graphical representations. The authors explained that some PSTs' actions suggested their enacting meanings were dominated by fragments of sensorimotor experience and perceptual properties of shape, while other PSTs' actions suggested their enacting meanings were dominated by the coordination of mental operations in the form of quantitative reasoning.

Table 4.2 Useful definitions

Construct	Definition	Other references
Quantity	“A quantity is a quality of something that one has conceived as admitting some measurement process” (Thompson, 1990, p. 5).	Smith III and Thompson (2007), Steffe, (1991b) and Thompson (1993)
Magnitude	“The idea of magnitude, at all levels, is grounded in the idea of a quantity’s size” (Thompson et al., 2014, p. 1). “Magnitude refers to the size or amount-ness of a quantity that remains invariant with respect to changes in the unit used to measure the quantity” (Moore et al., 2019b, p. 3).	Liang and Moore (2021)
Value (or measure)	“The numerical result of a quantification process applied to [a quantity]” (Thompson, 1990, p. 6).	Moore et al. (2013), Smith III and Thompson (2007), and Thompson (2011)
Number	We use “number” to refer to when a student is referencing a numerical signifier that is not the result of a quantification process applied to a quantity to obtain a measure.	Smith III and Thompson (2007) and Thompson et al. (1994)
Quantitative operations	We use quantitative operations to refer to both “the conception of two quantities being taken to produce a new quantity” (Thompson, 1990, p. 11) as well as the operations that generate quantity or are involved in measuring a quantity, such as partitioning, unitizing, and iterating (Steffe, 1991b).	Liang and Moore (2021), Steffe and Olive (2010) and Thompson (1994a)
Arithmetic or numerical operations	“[Operations] used to calculate a quantity’s value; there is no direct correspondence, except in a canonical sense, between quantitative operations and the arithmetic operations actually used to calculate a quantity’s value in a given situation” (Thompson, 1990, p. 12).	Ellis (2007a), Smith III and Thompson (2007) and Thompson et al. (1994)
Quantitative relationship	“A quantitative relationship is the conception of three quantities, two of which determine the third by a quantitative operation” (Thompson, 1990, p. 13).	Moore et al. (2013) and Thompson and Thompson (1996)
Quantitative reasoning	“Quantitative reasoning is the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships” (Thompson, 1990, p. 13).	Ellis (2007a), Moore (2014a) and Smith III and Thompson (2007)
Covariational reasoning	“...someone holding in mind a sustained image of two quantities values (magnitudes) simultaneously...one tracks either quantity’s value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value” (Saldanha & Thompson, 1998, p. 298).	Carlson et al. (2002), Johnson (2012) and Paoletti and Moore (2018)
Directional change	“Coordinating the direction of change of one variable with changes in the other variable” (Carlson et al., 2002, p. 357).	Moore (2014a), and Thompson and Carlson (2017)

(continued)

Table 4.2 (continued)

Construct	Definition	Other references
Amounts of change	“Coordinating the amount of change of one variable with changes in the other variable” (Carlson et al., 2002, p. 357).	Ellis et al. (2015), Johnson (2015) and Liang and Moore (2021)

Moore, Stevens, et al. (2019b) provide an apropos example of Thompson’s (1985) reframing because all PSTs in the study undoubtedly enacted mental operations when problem solving. Furthermore, they did so in the presence of available or produced figurative material, thus not having to regenerate material entirely from scratch. Accordingly, early distinctions of figurative and operative thought would not have been productive in providing differentiated accounts of their actions. By adopting Thompson’s framing, Moore, Stevens, et al. (2019b) differentiated whether the PSTs’ actions foregrounded properties of figurative material itself, including that which resulted from their activity, or whether their actions foregrounded the coordination of mental operations not necessarily tied to particular properties of figurative material. In all, they identified that some PSTs’ actions foregrounded figurative aspects of thought, including requiring starting a graph on the vertical axis, drawing or interpreting a graph left-to-right, drawing a graph that passes the “vertical line test,” and drawing a graph that maintains some template or recalled shape, to mention a few. On the other hand, they identified some PSTs’ actions that foreground operative aspects of thought in the form of quantitative and covariational reasoning so that their construction and interpretation of graphs were dominated by those operations no matter the resulting shape of their graph; the shape was entirely a consequence of their coordination and tracking of quantities’ variation.

As an illustrative example, Moore, Stevens, et al. (2019b) identified some PSTs’ meanings that involved their perceiving “slope” in terms of left, right, up, and down physical movements, and then associating paired movements with numerical properties for slope. For instance, one student had constructed the following (movement, slope) pairs: (left-up, negative), (right-down, negative), (left-down, positive), and (right-up, positive). Importantly, these (movement, slope) pairs were not dependent on axes’ orientations or labeling, but instead properties of a *line qua line*. Moore, Stevens, et al. (2019b) illustrated that such a meaning is viable when particular Cartesian conventions are maintained, but its foregrounding of particular sensorimotor movement results in significant perturbation when those conventions are not maintained.⁵ For instance, a PST in their study perceived the two displayed graphs in Fig. 4.1 to have a positive and negative “slope,” respectively. She was unable to reconcile to her satisfaction the perturbation that stemmed from each displayed graph having different “slopes,” yet each being associated with the same formula that implied a positive “slope” of 3.

⁵As Moore, Silverman, et al. (2019a) described, our use of “convention” here is not precise. What the researchers perceived as a convention was certainly not a convention to the referenced participants of the study.

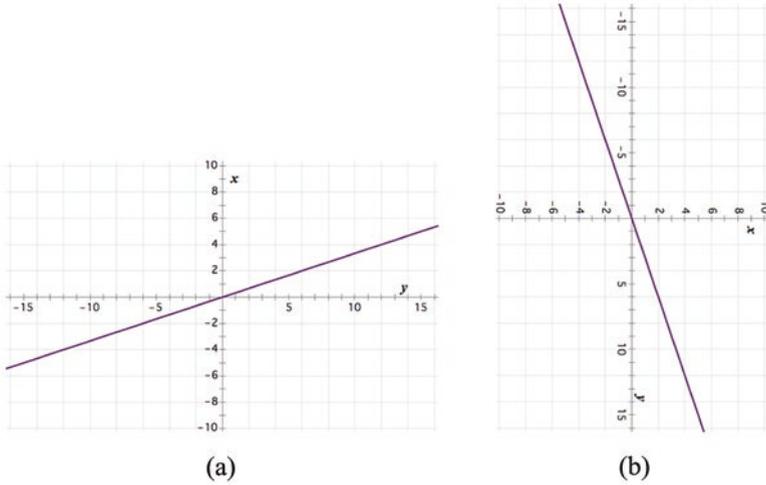


Fig. 4.1 Two displayed graphs of $y = 3x$, each with noncanonical axes orientations

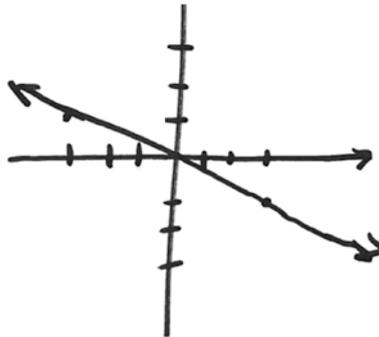


Fig. 4.2 A displayed graph of $y = 3x$ when scales are homogeneous; and y is oriented with positive values horizontally to the right and x is oriented with positive values vertically downward, or y is oriented with positive values horizontally to the left and x is oriented with positive values vertically upward

On the other hand, Moore, Stevens, et al. (2019b) illustrated other PSTs’ meanings involved their perceiving properties of “slope” as a consequence of rate of change and its representation under the constraints of a coordinate system. These PSTs thus associated slope with the coordination of quantities’ values and a comparison of amounts of change. Reflecting the generativity of operative forms of thought, such a meaning proved viable for the PSTs no matter the coordinate orientation, as their reasoning persistently foregrounded the coordination of quantitative operations so that perceptual properties of displayed graphs were always a consequence of those coordinations. For instance, a PST in their study conceived numerous axes-orientations and labeling to achieve a positive “slope” with the displayed

graph in Fig. 4.2. The PST's comment on a student who claims that the line has "a negative slope" further illustrates the foregrounding of quantitative and covariational reasoning (Excerpt 4.1).

Excerpt 4.1 Annika Coordinating Two Quantities' Values to Address a Student Claim Annika: You'd have to notice that even though it looks like a negative slope [*making a hand motion down and to the right*] because we call it slope because it's visual and it's easy to visualize a negative and positive slope [*making hand motions to indicate different slopes*]. But that's only visual on our conventions of how we set it up. Um, but like [*pointing to the graph*] if slope is rate of change we can still see that for like equal increases of x [*making hand motions to indicate equal magnitude increases*] we have an equal increase of y [*making hand motions to indicate equal magnitude increases*] of three. And so for equal positive increase of one [*sweeping fingers vertically downward to indicate an increase of one*], we have an equal positive increase of three [*sweeping fingers horizontally rightward to indicate an increase of three*]. And so it is a positive slope. (Moore et al., 2019b, p. 12)

Informing Generalized Models of Student Thinking

Whereas Moore, Stevens, et al. (2019b) focused on students' graphical meanings enacted in-the-moment of problem-solving, the distinctions between figurative and operative thought are also useful in articulating generalized models of students' graphical meanings. These generalized models, referred to as *epistemic subjects*, are conceptual models that specify categorical differences among students' meanings and ways of thinking (Steffe & Norton, 2014; Thompson, 2013a). Specifically, the distinctions between figurative and operative aspects of thought informed Moore and Thompson's constructs of *static (graphical) shape thinking* and *emergent (graphical) shape thinking*, respectively. Moore and Thompson (Moore, 2021; Moore & Thompson, 2015) introduced static shape thinking as a way to refer to meanings that involve conceiving a graph as if it is essentially a malleable piece of wire (graph-as-wire). They introduced emergent shape thinking as a way to refer to meanings that involve conceiving a graph as a trace entailing corresponding values (or magnitudes) that are produced through the covariation of quantities.

Examples of static shape thinking include iconic translations—associations between the visual features of a situation and those of a graph (Monk, 1992)—and thematic associations—associating properties of event phenomena with shape properties of a graph that are superfluous to the stated quantitative referents (Thompson, 2016). An example of an iconic translation is a student perceiving a graph as containing a circle because the situation entails a circle. An example of a thematic association is a student perceiving an object traveling at a constant speed as necessarily implying a line for a graph or an object traveling at a varying speed necessarily implying a curve for a graph, regardless of the quantities being graphed. Other examples of static shape thinking include facts of shape in which figurative

properties of a graph (or perceptual comparisons between graphs) imply particular equations, analytic rules, names, mathematical properties, and so on. For instance, an individual might associate a graph that curves up with an exponential relationship, or an individual might associate a line with a linear relationship regardless of the coordinate system. As another example, a student might associate the shape of a graph with only one analytic rule, which constrains their ability to accommodate novel graphing experiences (Moore, 2021). In each case (Fig. 4.3), the associations foreground perceptual properties of shape and associations based on those properties.

Examples of emergent shape thinking include any instance in which an individual conceives a displayed graph as the product of coordinating and tracking two or more varying quantities. This could occur in the context of a student conceiving a displayed graph as having emerged through tracking quantities' covariation, or it could occur in the context of a student producing a displayed graph as a trace that captures some conceived covariational relationship. The latter case is often for the purpose of producing a displayed graph that is a quantitative model of some situation or relationship held in mind. Producing such a displayed graph involves the student re-constructing the coordination of covarying quantities to produce a trace they perceive as capturing a covariational relationship equivalent to the relationship conceived in the situation or relationship held in mind (Fig. 4.4). Relatedly, a student can reason emergently to compare and contrast displayed graphs in terms of the covariational properties involved in producing their trace. For instance, a student can reason emergently to conceive two displayed graphs as representing an invariant covariational relationship despite perceptual differences between the graphs due to their being represented under different scales (Fig. 4.5), orientations (Fig. 4.1), or coordinate systems (Fig. 4.6) (Moore et al., 2013). Similarly, emergent shape thinking could involve a student conceiving a displayed graph as being produced by two different covariational relationships despite the result of each trace being perceptually equivalent (Fig. 4.7, left and right, each illustrating a progressive trace). In each case, any association a student makes with a graph is an implication of the covariational relationship they understand the displayed graph to have emerged as a trace of.

The relationships between Thompson's framing of figurative and operative thought and Moore and Thompson's graphical shape thinking constructs are apparent by the object of reasoning—a displayed graph—and its associations. With respect to static shape thinking, such a way of thinking foregrounds figurative aspects of thought; static shape thinking entails actions based in perceptual cues and properties of shape. A displayed graph is essentially an object in and of itself, and associations with the displayed graph are learned facts of the *shape qua shape*. Such associations are consistent with von Glasersfeld's (1987, 1995) discussion of figurative icons, symbols, or words. In the case of constructing a displayed graph, a student thinking statically carries out activity for the purpose of drawing a shape consistent with learned or memorized associations. Those learned or memorized associations are not conceived as organic or inherent to the production of the graph, but rather they are simply taken as assigned to a *produced* shape that is perceptually consistent with the association. The learned or memorized associations are abstractions based on the products of activity and particular states. Furthermore, because a

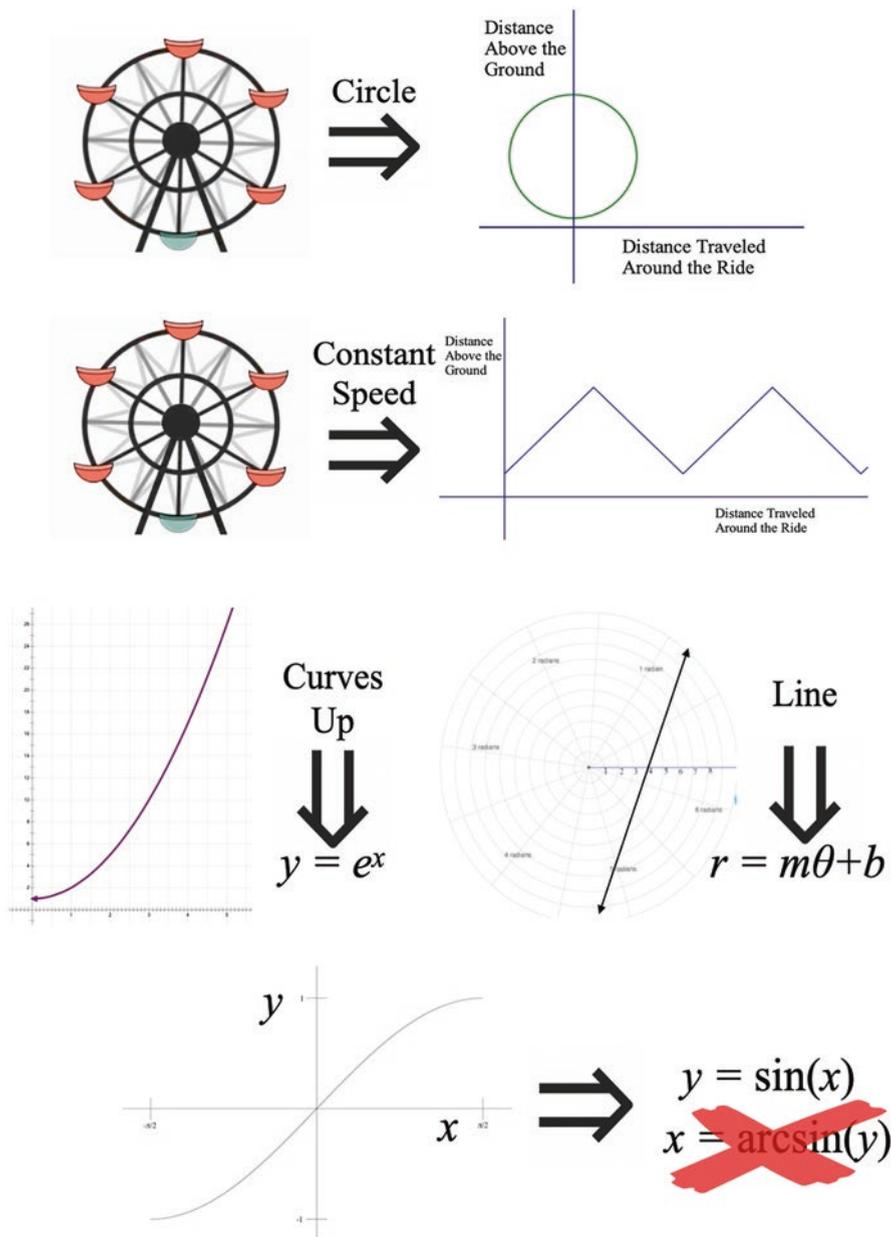


Fig. 4.3 Examples of static shape thinking and its implications

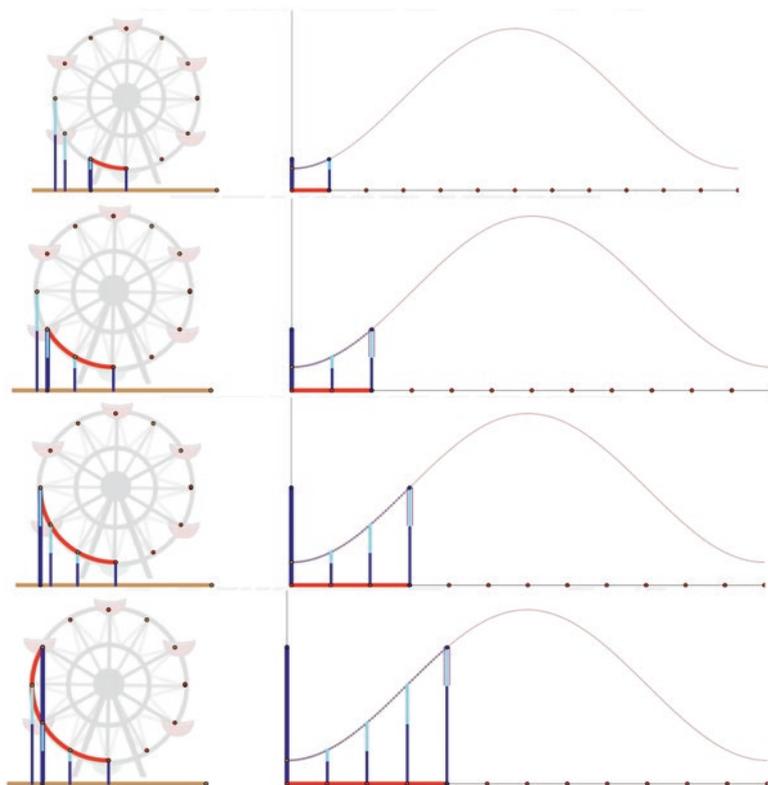


Fig. 4.4 Constructing a covariationally equivalent emergent trace

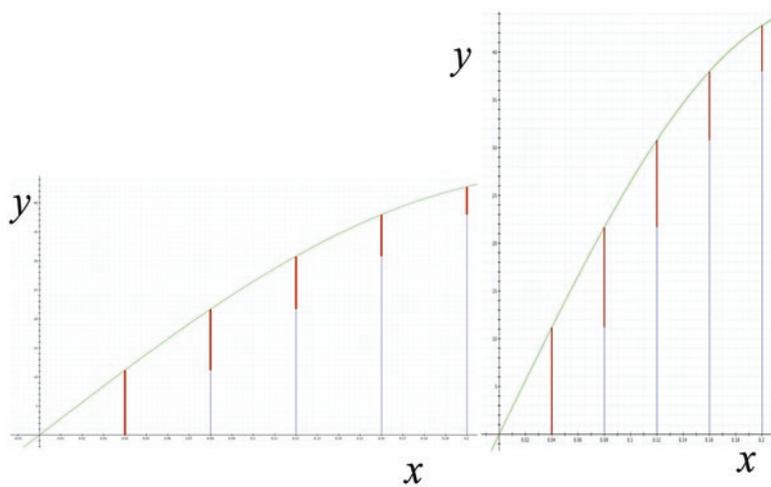


Fig. 4.5 Two displayed graphs of $y = 45 \sin(2\pi x)$

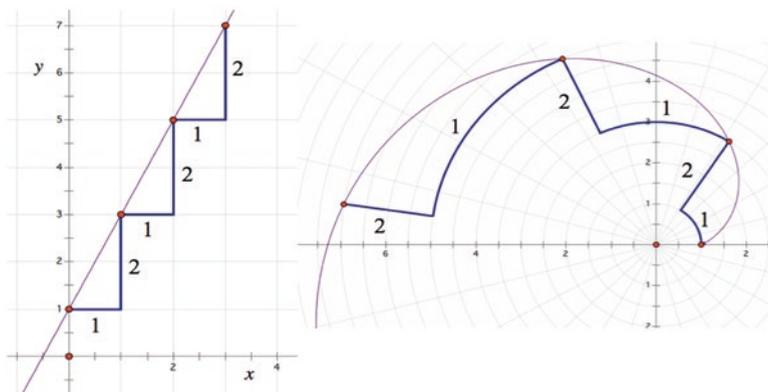


Fig. 4.6 Two displayed graphs equivalent to the relationship defined by $z = 2k + l$

student thinking statically foregrounds perceived properties of a produced or anticipated shape, they can reject a displayed graph generated via reasoning emergently when the resulting properties of the produced shape are incompatible with the student's static shape thinking (see Polly in Moore et al., 2019b).

With respect to emergent shape thinking, such a way of thinking foregrounds operative aspects of thought; emergent shape thinking entails the coordination of mental operations in the form of quantitative and covariational reasoning. A displayed graph is understood as reproducible through the coordination of the mental operations that resulted in its trace, and associations with the graph are anticipated as the mathematical properties organic to the operations that produce it. Any learned or memorized associations are understood as properties of or symbolizing the coordination of operations represented by the displayed graph. Such associations are consistent with von Glasersfeld's (1987, 1995) discussion of operative icons, symbols, or words. Furthermore, because a student's thinking emergently foregrounds the coordination of mental operations, their meanings are positioned to accommodate novel graphing experiences. The coordination of those mental operations and their perceived mathematical properties provide actions of thought that can be transformed into novel figurative material, all the while maintaining invariance in the logico-mathematical form. That is, emergent shape thinking enables the enactment of operations that can differ in terms of their figurative entailments yet be compared on the basis of similarities and differences in the mathematical properties of those operations.

Adapting the Distinctions to Other Representations

Thus far, we have illustrated examples of figurative and operative thought through graphical representations and phenomena. However, definitions of figurative and operative thought have been adapted in ways to address other representations. In

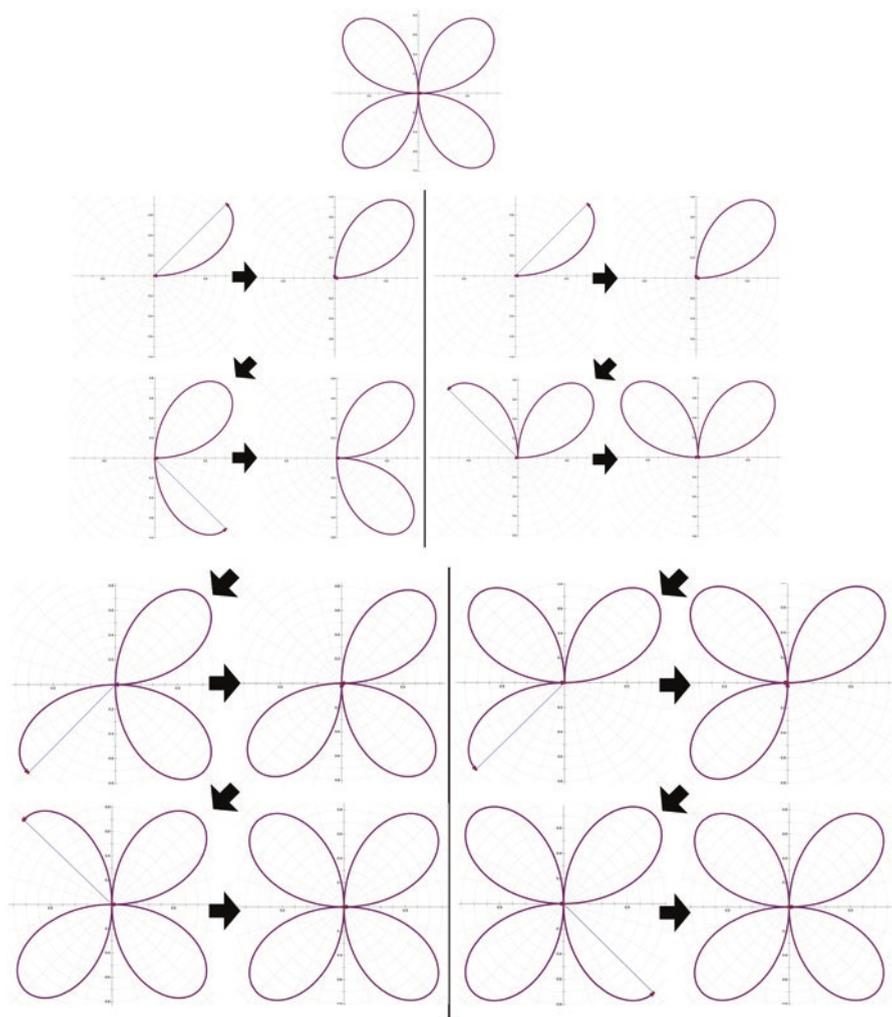


Fig. 4.7 One completed displayed graph, two covariational relationships (left, $r = \sin(2\theta)$; right, $r = |\sin(2\theta)|$)

this section, we consider these adaptations when discussing students' reasoning with formulas.

As a reminder, Thompson (1985, p. 195) said, "Any set of schemata can be characterized as figurative or operative, depending upon whether one is portraying it as background for its controlling schemata or as foreground for the schemata that it controls." In considering how to adapt the constructs of figurative and operative thought in a new representation, one needs to consider what mental operations are involved in students' acts of re-presentation. With formulas, a researcher characterizing an enacted meaning does not rely on perceptual actions taken on material,

such as tracing right to left as Moore, Stevens, et al. (2019b) described, or repeating partitioning activities across contexts as Liang and Moore (2021) described. Whereas graphical representations and phenomena provide material to engage operatively or figuratively, formulas merely provide symbols in the form of inscriptions or glyphs. Thus, a researcher characterizing a student's formula meanings as figurative or operative relies on the associations evoked by a symbol (or collection of symbols) in a formula.

In a teaching experiment in which Stevens (2019) attempted to understand and support students' meanings for formulas through reasoning with dynamic situations, she built upon research involving the Piagetian constructs of multiplicative objects and figurative and operative thought to model students' ways of thinking about graphs. Her strategy of the experiment focused on designing tasks that would enable her to construct models of students' mathematics by identifying opportunities for students to reason quantitatively and covariationally with formulas, and her analysis focused on characterizing students' mental actions in terms of the nature of associations evoked by given or constructed formulas. In particular, she adopted the constructs of figurative and operative thought to differentiate between associations that stemmed from perceptual similarities or learned facts and those that stemmed from enacting quantitative operations and abstracting their mathematical properties.

To discuss figurative thought in more detail, consider how students might answer the following prompt: "Describe a situation in which the formula $A=2\pi rh$ describes a relationship between quantities. How does your situation describe that relationship?" The meanings students have for the formula $A = 2\pi rh$ dictate the associations they will draw between a situation and the formula. Consistent with static shape thinking examples above, a meaning foregrounding figurative aspects of thought would foreground associations between attributes or shapes (e.g., area of circle, or surface area of a cylinder or spherical cap) and aspects of the symbols constituting the formula. These associations may or may not have quantitative entailments, but importantly those entailments are not inherent to the association. For example, a student may conceive the formula $A = 2\pi rh$ as the normative formula for the surface area of an open cylinder, and *only* an open cylinder (see Stevens (in press) for this example). As a second example, a student might also identify " 2π " as necessarily pointing to a feature of a circle and, thus, that the relevant situation must include a circle. As Stevens (in press) illustrated, a student stated that $A = 2\pi rh$ could not represent a formula associated with a parallelogram. They explained, "Because I don't know what it [2π] would represent in a parallelogram. Like, in a circle, it's because you can like divide a circle into two pi radii, but you don't have anything even here that you could do that with." In both examples, the reasoning foregrounds associations between the arrangement or presence of the glyphs in the formula and particular shapes. Thus, in the event that the associations do have quantitative entailments, those entailments are constrained to the objects of association.

Alternatively, operative thought would entail meanings that focus on the formula as representing quantitative relationships that could be re-presented in a variety of contexts. As an example, $A = 2\pi rh$ represents a linear relationship between height

and area that could be relevant to a rectangle, cylinder, or spherical cap, depending on what quantities the symbols represent. A student could thus generalize that $A = 2\pi rh$ means that for equal changes in height, the surface area of the relevant shape, whatever it is, increases by an amount that is always $2\pi r$ times as large as that change in height. Stevens (2019) described a student who reasoned that such a relationship was implied by the formula, and the student then generated that relationship using cylinder and spherical cap contexts. Figure 4.8 provides an illustration of this way of reasoning; the corresponding strips of the cylinder and spherical cap have equal surface area. Consistent with the notion of operative thought, the student's reasoning foregrounds quantitative relationships, and the coordination of quantitative operations drives their activity and products.

It is important to note that figurative associations, such as associating πr^2 with the area of a circle, can be useful in constructing a quantitative structure in a context. Similar to the notion of expert shape thinking (Moore & Thompson, 2015), as long as a student can unpack those associations quantitatively (similar to the reasoning illustrated in Fig. 4.8), while also anticipating that there can be other potential associations. Such associations reduce cognitive effort; it is inefficient to always enact quantitative operations.

It is also important to note the effects of foregrounding figurative associations in constructing formulas. A student who constructs figurative associations between geometric shapes and formulas in a way that dominates any quantitative entailments may attempt to identify perceptual features in the context and then attempt to incorporate formulas associated with those figurative elements in their construction of a formula, resulting in a nonquantitative formula. For instance, in Stevens (2019), a student, Kimberley, identified both a dynamic cone and cylinder as including circles and varying height. She concluded that only one should be able to be associated with the formula $A = 2\pi rh$. She eliminated the cone by foregrounding figurative aspects of thought and breaking the cone into particular shapes that she could then associate with area formulas. As a result, Kimberley's formula for the cone combined her formulas for the area of the circle with the area of a triangle (Fig. 4.9).

In summary, although mathematics education researchers have primarily focused the figurative and operative distinctions on representations or phenomena in which students can enact quantitative operations, the distinctions are also applicable to the

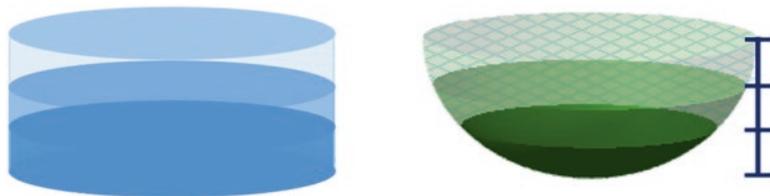


Fig. 4.8 An example of operative thought with $A = 2\pi rh$ by considering A and h as variables to conclude that for equal changes in height, there are equal changes in surface area for cylinder and spherical cap (A = surface area, r = radius length of circle/sphere, h = vertical height)

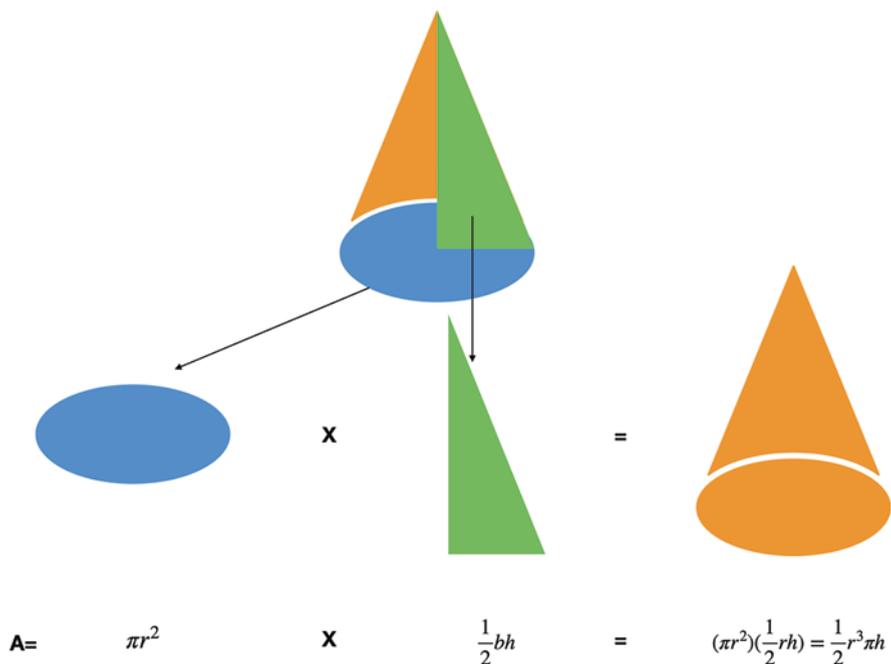


Fig. 4.9 Example of figurative thought used to construct formula for surface area of a cone by multiplying together shapes in geometric figure

associations students construct for symbols, groups of symbols, or formulas. Furthermore, such associations are consistent with von Glasersfeld’s (1987, 1995) broader discussion of icons or symbols with attention to the extent a formula evokes a learned association or a set of anticipated quantitative operations. Incorporating a focus on formulas also helps illustrate that although associations that are figurative in nature (e.g., $A = \pi r^2$ provides a way to calculate the area of a circle) are beneficial for reducing cognitive load, it is important that such associations are not constructed at the loss of formulas representing a quantitative relationship specific to a context or particular object (e.g., $A = \pi r^2$ representing that the area of a circle increases by increasing amounts as the radius increases by a constant, successive amount). Furthermore, it is important that such operative associations are not constructed at the loss of a formula potentially representing a variety of quantitative relationships (e.g., $A = \pi r^2$ is the area of a rectangle with a side π times as large as the other side). In summary, operative meanings that foreground symbols and groups of symbols in formulas as pointers to quantities, quantitative operations, and covariational operations are generative in that they enable a student to conceive formulas as relevant to both experienced and unexperienced situations, all the while understanding the formula captures some invariance property across those situations.

Transitioning the Constructs Back to the Study of Meaning Construction

The examples in the prior section used figurative and operative distinctions when discussing undergraduate students' reasoning with representations (e.g., graphs and formulas). The aforementioned work occurred after students had constructed particular meanings for those representations through schooling; the studies did not address students' initial construction of graphical meanings. In this section, we illustrate the extension of these constructs to the construction of graphs, which was led by Tasova (2021). Tasova (2021) adopted figurative and operative thought to characterize middle-school students' graphing meanings as they engaged in solving problems involving the construction of graphical representations of varying quantities for what appeared to be the first time (at least within formal settings). In doing so, he illustrated the viability of explaining middle-school students' reasoning with the figurative and operative distinction, while also adapting those distinctions to explain his data. Ultimately, he identified meanings compatible with figurative or operative aspects of thought. Underscoring von Glasersfeld's (1995) framing that operations always have to operate on something, Tasova also identified important meanings that entailed aspects of both figurative *and* operative thought in the same activity. We summarize those findings here.

Before supporting middle schoolers in developing meanings for graphs in two-dimensional space, Tasova (2021) provided them opportunities to engage with quantities' magnitudes represented by varying lengths of directed bars placed on empty number lines (also called *magnitude lines*, see Fig. 4.10, right). Students had opportunities to use these dynamic segments to represent individual quantities in one-dimensional space. As one example, students played the bike animation (see Fig. 4.10, left) and pulled the blue segment in ways that could represent how the bike's distance from Arch (DfA) varied in the situation.

Tasova (2021) identified two meanings for the dynamic segments that align with figurative and operative aspects of thought. An individual who conceives the bike's DfA in the situation as decreasing and increasing might vary the length of the segment on the magnitude line accordingly. This is an example of an operative aspect of thought as the individual disembeds the conceived quantity's magnitude from the situation, re-presents it on the magnitude line, and then coordinates the length of the



Fig. 4.10 Downtown Athens Bike Task and magnitude line

blue bar, ensuring to preserve its length as the animation played. As an alternative conception, an individual can simulate the physical bike on the magnitude line. For example, they might coordinate the direction they “pull” one end of the blue segment with the bike’s right and left movements, thus associating the movement of the bike with the movement of the segment endpoint. Tasova identified this as an example of a figurative aspect of thought as the student’s thought is dominated by imitating the moving object in the situation.

As a second example of using the figurative and operative distinction to characterize a nuanced difference in middle-school students’ actions, we use another case that involves re-presenting a quantity on the magnitude line. In this case, the bike in the animation and the dynamic segment on the magnitude line are synchronized—the length of the blue bar on the magnitude line varies according to the bike’s movement on the map by design (see the animation at the following link: <https://youtu.be/6kdbDeVEF9w>). For example, while moving the bike to the right from its position (Fig. 4.10, left), the right end side of the blue bar on the magnitude line moves to the left (indicating the bike’s DfA is decreasing). This is different from the previous task in that the endpoint moves in the opposite direction of the bike.

Explaining how this could happen, a seventh-grade student referred to the map and claimed, “The bike is getting closer to Arch.” Then, by pointing to the blue bar on the magnitude line and tracing the pen over the line from right to left, the student said it “is gonna get closer to right here [*pointing to the zero point on the magnitude line*], which is Arch.” Tasova argued that the student was not conceiving the bar as a varying length, but instead that the student conceived of the Arch and bike as physically placed on the left and right endpoints of the blue bar, respectively. He thus perceived the blue bar’s change to be a product of the bike moving closer in proximity to the Arch along the blue bar. Tasova contrasted this with a response from another student who first determined that the bike’s DfA is decreasing, while moving the bike to the right on the map. The student then conceived the length of the blue bar on the magnitude line as a re-presentation of the bike’s DfA, thus requiring that the length decrease to remain equivalent to the length in the situation. She explained, “it [*pointing to the blue bar*] is gonna get smaller because distance is smaller on the number line too.” Moreover, she labeled the starting point as “zero,” whereas the previous student conceived the same point on the magnitude line as “Arch.”

Reflecting on the two students’ actions, they are compatible in that they each had the same result from an observer’s perspective: the students moved the blue bar endpoint so that it decreased in length. But their meanings were notably different. In the first case, and consistent with reasoning that foregrounds figurative notions of thought, the meaning foregrounded the spatial proximity of two objects and moving them accordingly. Any change in the blue bar came as a consequence of changing their spatial proximity. In the second case, and consistent with reasoning that foregrounds operative notions of thought, the meaning foregrounded the distance between two objects. The length of the blue bar was persistently in mind, and the student’s goal was to maintain a length equivalent to that in the situation. Although these differences are subtle, Tasova (2021) illustrated they have significant

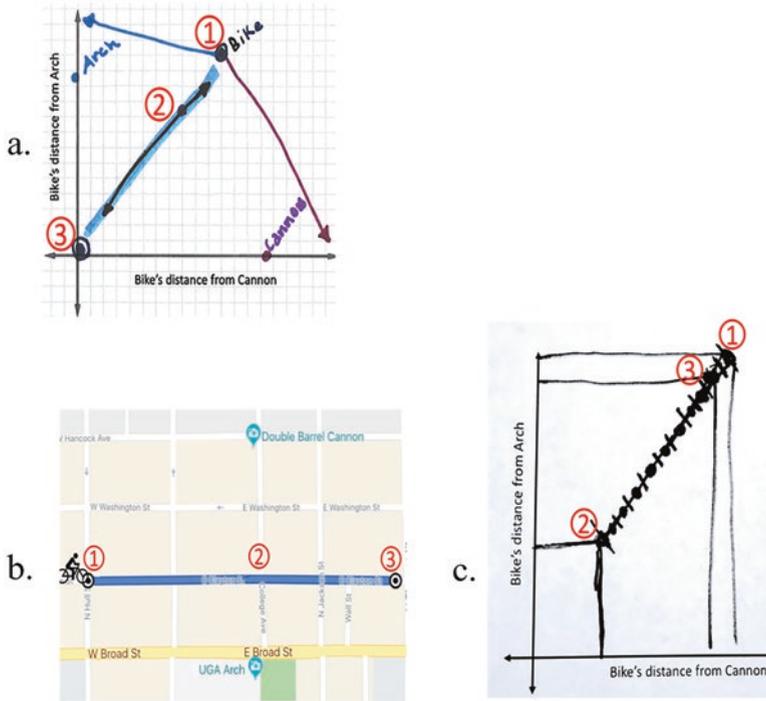


Fig. 4.11 (a) DABT, (b) Zane’s graph, and (c) Melvin’s graph

consequences for students’ graphing actions. To illustrate, we use two middle-school students’ activity during the Downtown Athens Bike Task (DABT).

DABT prompts students to use a Cartesian coordinate system to draw a sketch of the relationship between the bike’s DfA and the bike’s distance from Cannon (DfC) as the bike moves on Clayton St. in Downtown Athens (see Fig. 4.11a). The bike starts from the West side of the street and moves at a constant speed. To illustrate an example of figurative thought, we draw on Zane’s (a seventh grader) activity in which his graphical meanings representing literal pictures of a particular situation. Zane conceived the Arch and Cannon placed at physical locations on the vertical and horizontal axes, respectively, as implied by the labels (see blue and purple dots on each axis for Arch and Cannon, respectively, in Fig. 4.11b). Because Arch is at the top and Cannon is at the bottom of the plane, Zane then imagined rotating the plane in a way that Arch is at the bottom and Cannon is at the top, just as with the map. He then drew a segment on the plane, identifying it as the path of the bike. He also added dots on the plane (i.e., one that he labeled “Bike” and the other at the origin), which he placed to match the dots at each end side of the bike’s path on the map (see Fig. 4.11a). Zane went to the extent of using the same coloring for the segment on the plane as that on the map (i.e., light blue). Drawing on the notion of iconic translation (Clement, 1989; Monk, 1992), Tasova (2021) characterized

Zane's meaning as *transformed iconic translation*. Zane translated a transformed version of perceptual features of the situation to the plane as he rotated the plane and overlaid it into the map in order to perceptually match the graph that he drew on the plane with the bike's path on the map.

Whereas Zane's example is a rather typical example of a graphical meaning that foregrounds figurative aspects of thought, Melvin's activity in DABT provides a novel illustration of how an individual's meaning might involve both figurative and operative aspects of thought. Melvin conceived the entire vertical and horizontal axes of the plane as the physical Cannon and Arch, respectively. The axes did not represent magnitude or number lines, but instead were referents to the objects themselves. Melvin then drew a line upward from left to right to represent "where the bike travels." He added tick marks and dots on his line graph "to represent like where the bike could be." For example, when the bike is at location 2 in the map (i.e., DfA and DfC are each at their minimum), he said, "It [the bike] would go right here [*pointing to the tick mark on his graph near 2 in Fig. 4.11c*]." Similar to Zane, and consistent with reasoning that foregrounds figurative aspects of thought, Melvin conceived each point on his graph as the physical bike moving on its path on the plane.

Differing from Zane, Melvin also imagined moving the bike on the plane according to the variation of its DfA and DfC. He conceived DfC and DfA as being represented by vertical and horizontal segments drawn on the plane, respectively. The vertical segments on the plane represented the bike's DfC because Melvin assimilated the horizontal axis as a reference ray that he measured the bike's DfC from. Similarly, the horizontal segments on the plane represented the bike's DfA because he assimilated the vertical axis as a reference ray he measured the bike's DfA from. Increasing length of the horizontal and vertical segments on the plane indicated an increase in the bike's DfA and DfC. Moreover, Melvin understood that the length of the horizontal segment is shorter than the length of the vertical segment for each point on the graph as the bike's DfA is always less than the bike's DfC on the map. Thus, although Melvin assimilated his graph as "where the bike travels," Melvin's graph was not the bike's path as it is seen on the map; it was not an iconic translation. Melvin's meaning of the points included determining quantitative features of the bike in the situation (i.e., its DfA and its DfC), and he ensured they preserved those quantitative properties on the plane.

We consider Melvin's graphing activity as a different way of graphing relationships because his initial activity foregrounded figurative aspects of thought, but he then engaged in quantitative operations to make sense of the results of that initial activity and produce a graph to his satisfaction. Specifically, he represented the quantities' magnitudes in the space by committing to two frames of reference (i.e., the axes), and then placing a point where those magnitudes meet on the path he produced. Tasova and Moore (2020) termed this meaning of points, which has both operative and figurative entailments, as a *spatial-quantitative multiplicative object*, which involves an individual envisioning a point on the plane as a location/object that also entails quantitative properties. This example underscores that the figurative and operative distinction is not merely binary with respect to students' meanings.

As we discuss below, we believe this meaning, along with other meanings Tasova identified, illustrates the need to better understand how figurative and operative notions of thought play a role in students' construction of meanings that serve as both immediately and longitudinally productive.

Implications for Methodology and Task-Design

In addition to providing tools to construct second-order models of students' mathematics, as illustrated above, we have, in turn, found the figurative and operative distinctions useful in designing empirical studies. Specifically, we have found the distinctions useful in designing tasks and framing evidence for students' quantitative and covariational reasoning. Their usefulness in this regard can be organized by Moore et al. (2022) notion of an *abstracted quantitative structure (AQS)*, which the authors introduced as criteria for concept construction from a quantitative reasoning perspective. They described an AQS as a system of quantitative operations that an individual has interiorized so that it:

- C1. is recurrently usable beyond its initial experiential construction;
- C2. can be re-presented in the absence of available figurative material including that in which it was initially constructed;
- C3. can be transformed to accommodate to novel contexts permitting the associated quantitative operations;
- C4. is anticipated as re-presentable in any figurative material that permits the associated quantitative operations. (Moore et al., 2022).

The criteria were directly informed by the figurative and operative distinctions (Moore et al., 2022). C2 reflects the distinction between figurative and operative counting schemes as discussed by Steffe and colleagues (Steffe & Olive, 2010), as well as Glaserfeld's (1995) distinction between recognition and re-presentation. C3 captures the distinction introduced by Thompson (1985), allowing for the presence of figurative material and resting on the extent an individual is able to transform their actions to account for novel experiences and associated figurative material. C4 further extends C2 and C3 in a way that combines aspects of both Steffe's and Thompson's framings. Reflecting Steffe's framing, C4 involves an individual anticipating particular quantitative operations and their mathematical properties in the absence of figurative material. Reflecting Thompson's framing, C4 also involves the individual understanding that those operations and their mathematical properties are generalizable to any instance in which those operations are viable, thus allowing the individual to anticipate their enactment in the presence of novel figurative material. Such anticipation enables us to attribute particular quantitative and covariational relationships to quantities and figurative material too complex for us to enact those operations within. For instance, temperature is far too quantitatively complex for most individuals to enact particular quantitative operations, but we can anticipate variations in temperature and use proxies (e.g., values, number

lines, and coordinate systems) to represent those variations in a way that enable us to enact quantitative operations as if we are reasoning about temperature.⁶

As mentioned above, Moore et al.'s (2022) main intention of introducing the construct of an AQS was to provide guiding criteria for concept construction, but it also provides a framing for designing empirical studies and situating researcher claims regarding students' quantitative and covariational reasoning. Said succinctly, a researcher's claims regarding a student's quantitative and covariational reasoning are necessarily constrained by the nature of the tasks they use with a student.⁷ To illustrate this, we discuss each framing criteria above with respect to task design and potential claims regarding students' quantitative and covariational reasoning.

C1 refers to a quantitative structure that is recurrently usable beyond its initial experiential construction. From a researcher's perspective, and when considered independently of C2–C4, C1 is the most trivial of the criteria because it refers to an individual re-enacting a previously constructed quantitative structure in the presence of previously experienced context or figurative material. Thus, from a task-design perspective, this involves engaging the student in a task the student has already experienced and enacted the relevant quantitative operations. Here, "the same task" need not mean a carbon-copy of the previous task. Rather, "the same task" encompasses the figurative material provided and the quantitative operations needed to solve the task in the way the researcher hypothesizes. The researcher might change aspects of the tasks they consider to be surface-level features like values (e.g., the numbers being summed in a counting task) based on their model of the student's mathematics and what might be perceived as the same from the student's perspective.

A contraindication of C1—the student not being able to re-enact particular operations in the presence of an identical task—is evidence of the absence of quantitative or covariational reasoning. We do not consider indications of students having constructed an AQS consistent with C1 to enable strong claims regarding a student reasoning quantitatively or covariationally. Indications of C1 can be produced by a student mimicking or reproducing memorized actions from previous experiences. In such cases, a student is able to rely on the available figurative material and recall traces of activity from previous experience to reproduce previous results. However, to the researcher, the observable actions of such a student might be indistinguishable from those of a student enacting quantitative operations. It is thus necessary to design tasks that enable differentiating from a student mimicking or reproducing

⁶This underscores one of the more critical roles graphing plays in mathematics. Coordinate systems and their graphs provide quantities (e.g., oriented lengths or angle-openness) of which individuals can enact quantitative operations, thus serving as useful proxies for those quantities on which we cannot enact quantitative operations.

⁷It is more accurate and compatible with our theoretical perspective to say that a researcher's claims regarding a student's quantitative and covariational reasoning is necessarily constrained by the observable actions of the student's engagement. For the purposes of this chapter, it is more straightforward to discuss tasks as if they have some inherent or objective design feature, but we remind the reader that no such features exist and a task is always defined by the student's engagement.

memorized actions from a student whose reasoning foregrounds quantitative operations, which leads us to C2.

Much like C1, C2 is based on a student being able to re-enact particular quantitative operations. Unlike C1, the re-enactment comes in the absence of all or some subset of figurative material. With respect to a counting situation that involves an individual summing two collections of stones, the researcher might adjust the task so that one collection or both collections of stones may be hidden from the student with them only being told the numerosity of the hidden collections. As another example drawn from our own work, after a student constructs a particular relationship (e.g., the sine relationship) in the context of a Ferris wheel rider (Fig. 4.12), the researcher might prompt the student to re-construct that relationship with no figurative material provided (Fig. 4.13). Or, the researcher might provide a subset of figurative material as we did with the *Which One?* task in which a participant is asked to choose which red segments, if any, appropriately represent the height above the center (Liang & Moore, 2021, p. 300) (Fig. 4.14, also see the following link: <https://youtu.be/2pVVG18eEr0>). Note that these collections of tasks vary with respect to what is perceptually available to the student and what the student might be asked to mentally (or physically) generate, thus supporting a researcher in characterizing a student's re-presentation capacity.

In the *Which One?* task, we removed the Ferris wheel spokes because those spokes provide figurative material that can assist with partitioning the rider's trip

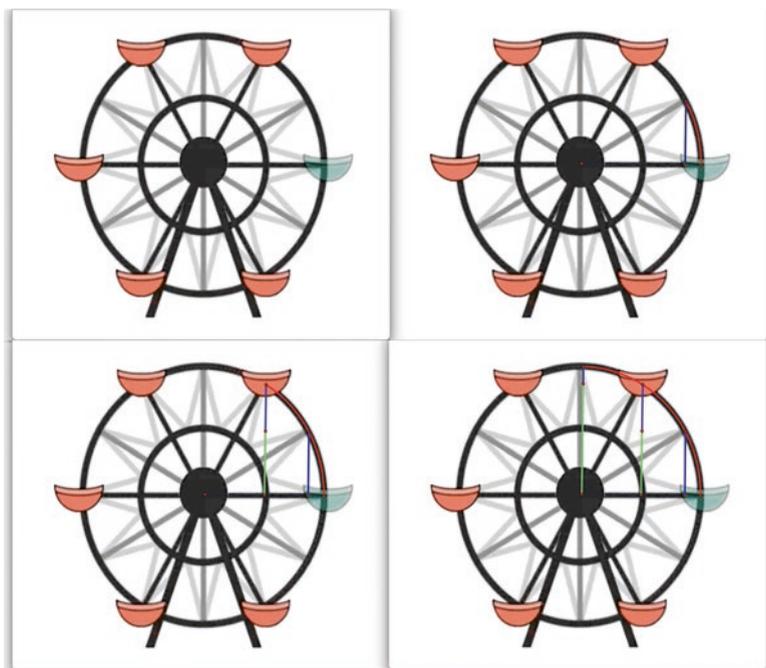


Fig. 4.12 Amounts of change and the sine relationship for $\pi/2$ radians of rotated arc

A Ferris wheel rider boards the ride at 3 o'clock. The ride begins to rotate counter-clockwise. How does the arc rotated by the rider vary with the rider's vertical distance from their starting place?

Fig. 4.13 A task with no provided figurative material

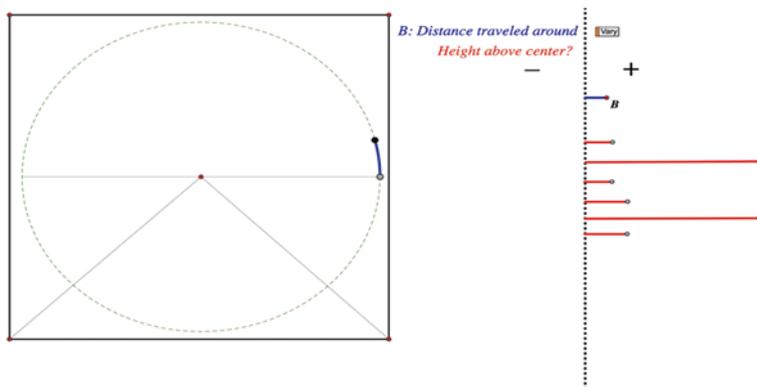


Fig. 4.14 The *Which One?* task environment as presented to students (Liang & Moore, 2021, p. 300), in which they are prompted to determine which red segments, if any, represent the point's height above the center of the Ferris wheel in relation to the arc length traversed by the point

into equal increases in arc length and corresponding variations in height. We also designed the task to include a digital environment in which they did have the availability of red segments, and these red segments could be re-oriented and embedded in the circle if the participant chose to do so. However, we designed the task so that the participant could not physically draw within the environment. Thus, as they moved the point around the circle, the chosen red segment varied accordingly, either in its given position or in a re-oriented position if they chose to do so. This required that the participant be able to hold in mind different states of the red segment and mentally produce amounts of change in that segment without the assistance of constantly available figurative material (Fig. 4.15).⁸

A contraindication of C2—the student not being able to re-enact particular operations in the absence of figurative material—is evidence that their meanings are tied to particular figurative material and carrying out activity tied to perceptual properties of that material. On the other hand, indications of students having constructed an AQS consistent with C2 enables a researcher to make relatively stronger claims

⁸The reader might note that the correct red segment can be determined solely through identifying that it is the correct length at each state. If a student bases their choice by checking states, we ask “Show us how to see the amounts of change in the red segment?” in order to prompt them to describe or generate partitions illustrating appropriate amounts of change.

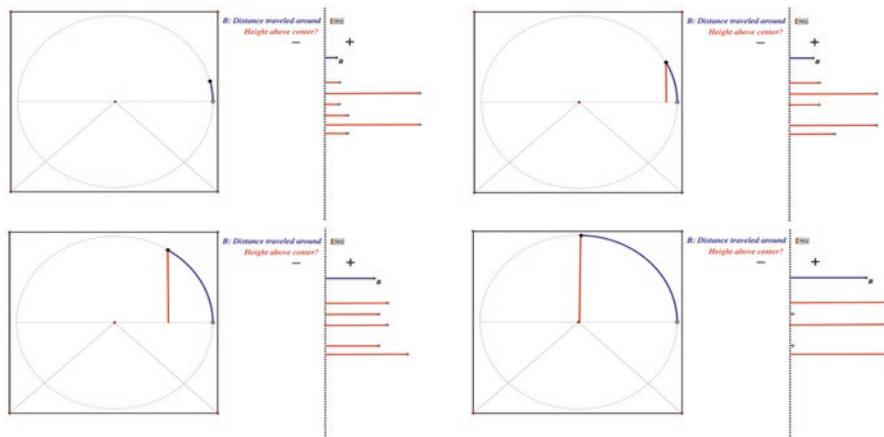


Fig. 4.15 The *Which One?* task and re-positioning a red segment as the object moves around the circle

regarding a student reasoning quantitatively or covariationally. An indication of C2 provides some evidence that the individual is able to re-enact quantitative operations to generate or anticipate figurative material that is no longer available. However, like C1, in our experience, students are able to provide evidence of C2 by what amounts to mimicking previous activity in order to produce absent figurative material. The production of that material often provides them source material for reflection and comparison to recalled previous activity and its results. Although the individual might enact quantitative operations in reflection and with the assistance of the material they have produced, this is not consistent with C2 as the operations enacted in reflection were not driving the production of the figurative material.

Whereas C2 involves an individual re-enacting quantitative operations in contexts previously experienced, C3 captures instances of an individual accommodating previously enacted quantitative operations to account for novel contexts. Much like a researcher determining what counts as “the same task” is dependent on their model of a student’s mathematics, what a researcher determines to be “a novel context” is also dependent on their model of a student’s mathematics. A novel context to a student could be a new phenomenon, a graphical representation, a graphical representation under a different coordinate orientation, or a graphical representation under a different coordinate system, to name a few (Fig. 4.16) (see the following studies for a collection of tasks designed with C3 in mind: Liang & Moore, 2021; Moore et al., 2013, 2019a, b; Paoletti et al., 2018). In some instances, such as changing a phenomenon or graphical coordinate system, what is novel may be the phenomenon and quantities under consideration. In other instances, such as considering different coordinate orientations, what is novel may be the figurative orientation of the quantities due to a change of referent. Regardless, a task designed with C3 in

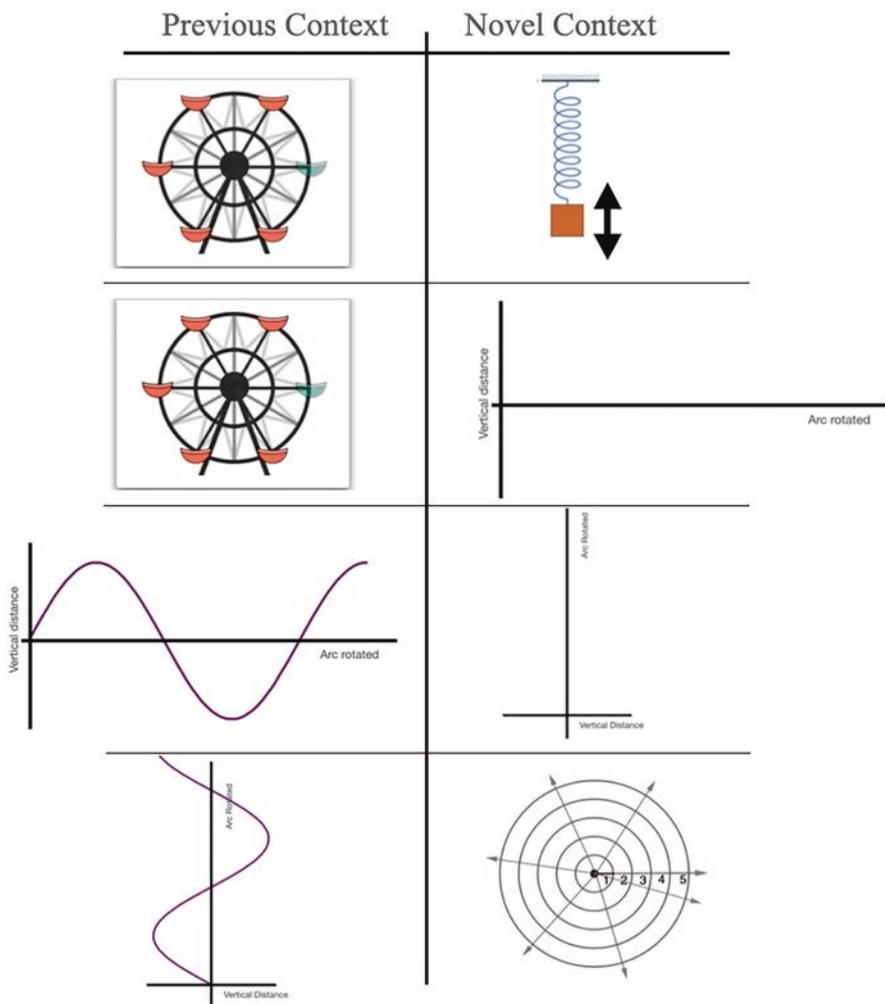


Fig. 4.16 Various task-design options in order to transition from a previously experienced context to a potentially novel context

mind affords the student the opportunity to re-enact previously enacted quantitative operations in the presence of different quantities or figurative orientations with the goal of perceiving an invariance across the differing contexts.⁹

⁹We note that C2 can be incorporated when designing tasks with C3 in mind. As an example, a researcher might intend to gain insights into whether a student can conceive a graphical representation as quantitatively equivalent to some relationship they conceive in a phenomenon. Taking C2 into account, the researcher might ask the student to construct that graphical representation, or they might provide figurative material in the form of displayed graph or graphs and prompt the student to determine which graph(s) accurately represent the relationship.

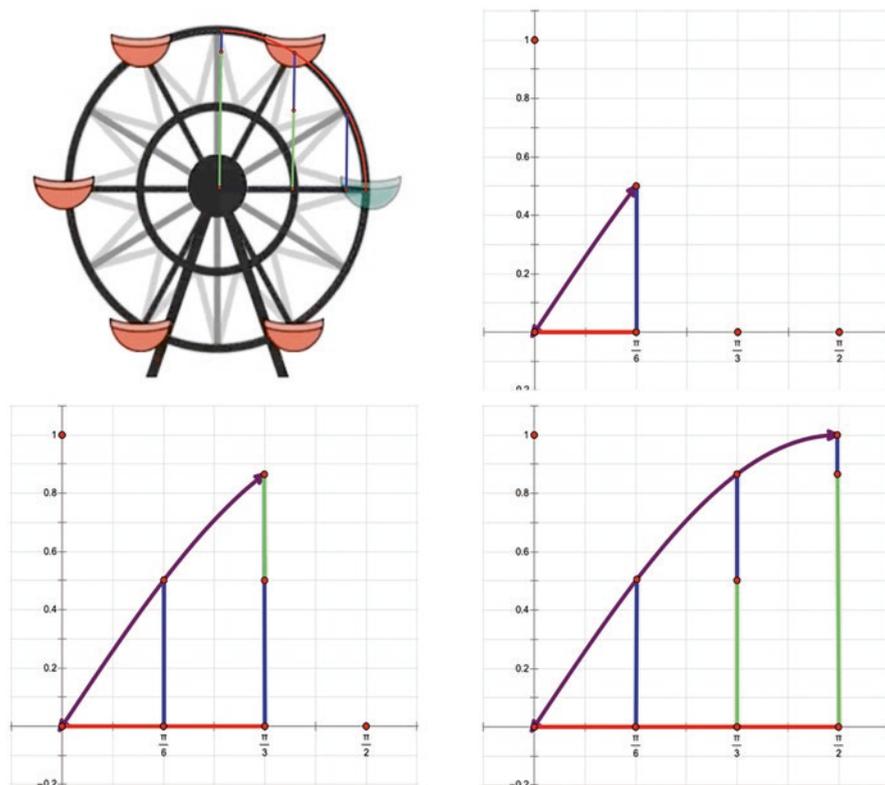


Fig. 4.17 A displayed graph emerging in a way that is invariant with that of the phenomenon

A hallmark of operative thought, per Thompson's (1985) framing, is that the coordination of mental operations and their transformation dominate an individual's activity. For that reason, indications of C3 provide a researcher with a strong evidentiary basis for claiming a student's reasoning foregrounds quantitative and covariational reasoning. Due to figurative differences across previously experienced and novel contexts, a researcher implementing tasks with C3 in mind is positioned to gain insights into whether a student can conceive quantitative invariance across those contexts despite figurative differences in their actions (Fig. 4.17). A researcher is simultaneously positioned to gain insights into when a student is mimicking previous activity. In instances in which a student provides a contraindication of C3, it suggests that there is some aspect of their meaning tied to figurative aspects of the contexts in which they previously enacted that meaning. This is not to say that the student's reasoning in those prior contexts was not quantitative, but rather that some aspect of their reasoning was reliant on particular figurative features of that context

and they thus looked to maintain those figurative features. For instance, in transitioning from a phenomenon like a Ferris wheel ride, the student might produce a graph from memory and then try to maintain partitions along a curve because there are partitions along a curve in the phenomenon (Fig. 4.18). Here, the graph does not emerge as a product of quantitative operations, but rather is produced to provide material to mimic activity or enact operations. As we discuss in the following section, students' attempts to maintain both quantitative and figurative aspects of their actions often lead to experienced perturbations due to incompatibilities between those maintained aspects.

Because C4 depends on an individual's capacity to anticipate some structure of quantitative operations, designing tasks incorporating this principle is complex. Furthermore, our research team has not concentrated on designing such tasks to the extent we have with that of C1–C3. In working with PSTs, one fruitful approach is designing tasks to include hypothetical student work that is nonnormative. Doing so provides a researcher insight into whether or not the PST anticipates that the hypothetical student might have produced the work through the viable enactment of the relevant operations. In the case that the PST does anticipate such an event, they may then attempt to determine that viable enactment (i.e., C3). In the case that the PST does not anticipate such an event, the PST is likely to enact meanings that identify some incorrect element of the student's work. Annika's response to Fig. 4.2 and determining numerous viable ways the figure is a displayed graph of $y = 3x$ is an example of the former. An example of the latter would be a PST immediately rejecting Fig. 4.1b as a potential displayed graph of $y = 3x$ because of its sloping downward left-to-right, or their rejecting Fig. 4.1a because the axes are incorrect (see Moore et al., 2019a for other examples).

Because C4 itself does not entail the enactment of quantitative operations, we do not consider indications of it to be evidence for quantitative or covariational reasoning. Rather, indications of C4 suggest that the individual has abstracted meanings that foreground quantitative and covariational reasoning to the extent that the relevant system of operations has become a way of thinking (Harel, 2008a, b; Thompson

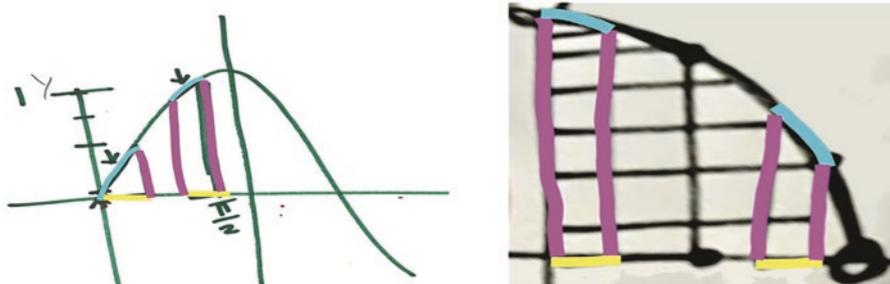


Fig. 4.18 Student work in which they drew a graph from memory and then attempted to reproduce partitions that maintained orientations and placements along segments or curves (Liang & Moore, 2021, p. 306)

et al., 2014). On the other hand, contraindications of C4 imply the individual has abstracted meanings that could entail quantitative and covariational reasoning, but they are such that they foreground some other aspect or form of thought. As the individual then enacts that meaning to make sense of the task, the researcher gains insights into the operations and associations constituting that meaning.

Moving Forward

The progress of mathematics education research is contingent on researchers pushing into new areas. In some cases, such pursuits require adopting or developing new theoretical perspectives or constructs. In other cases, such pursuits involve testing and adapting available theoretical perspectives or constructs. The work described in the previous sections is an example of the latter. Such an orientation toward theory stresses that theoretical perspectives and constructs should be pushed into new areas of study in an attempt to put that theory to the test. Doing so enables testing the viability of the theory and expanding its generalizability through adaptations that respond to constraints experienced in applying or developing the theory. Thus, our general suggestion for future work is that mathematics education researchers look to test the viability of the figurative and operative distinction within whatever area is of interest to them. Relatedly, we underscore that in pursuing such work, researchers keep in mind that they should not merely look to apply the distinctions as other researchers have, but rather view the notions of figurative and operative thought as malleable to their needs and experiences carrying out empirical work. Synthesizing the opening quotes by Dewey and Thompson, the weight of theory is in its ability to aid researchers' pursuits of problems and explanations of phenomena.

With respect to specific suggestions for research, we provide two potentially productive avenues. Our first suggestion is to continue the extensions taken by Tasova (2021) and Stevens (2019) to investigate the ways in which the figurative and operative distinction is relevant to students' construction of mathematical ideas and other representational systems. We envision at least two pivotal insights such research might provide. Research along the suggested lines will provide insights into how and the extent to which students construct meanings that transition from foregrounding figurative aspects of thought to those meanings in which operative aspects of thought dominate figurative entailments. Relatedly, such research will provide insights into those experiences that support or inhibit students' construction of meanings that foreground either figurative or operative notions of thought.

Research along the suggested lines might also provide insights into individuals' construction of incompatible or *competing meanings*. Tasova (2021) introduced the term *competing meanings* when expanding on Moore, Stevens, et al.'s (2019b) observation of a PST (Polly) having enacted two graphical meanings, one that foregrounded operative notions of thought and one that foregrounded figurative notions

of thought. In the case of Moore, Stevens, et al. (2019b), the PST perceived her enacted meanings to be incompatible with each other, which generated a perturbation and led to the PST rejecting the displayed graph she had constructed via reasoning emergently due to her having drawn the graph from left-to-right; she claimed that drawing a graph that way was “backwards.” Similarly, in his work with middle-school students, Tasova (2021) noted that students enact multiple graphical meanings. In some cases, the enacted meanings were incompatible with each other, leading to a student experiencing a perturbation and ultimately having to make an accommodation to their meanings or abandon one of their meanings. In other cases, a student’s enacted meanings were not necessarily incompatible with each other, but instead, each provided different avenues to solving the perceived problem. Regardless, Tasova’s (2021) notion of competing meanings generates a few questions for research that could benefit from incorporating the figurative and operative distinction: What are the different meanings students construct for a particular topic, and to what extent are those meanings compatible or incompatible? To what extent are particular competing meanings epistemological obstacles (Harel & Sowder, 2005) or natural to cognitive development, as opposed to artifacts of canonical educational approaches? How might competing meanings be productively addressed and leveraged by educators and researchers? With respect to this last question, there is the potential to generate productive learning experiences via students reflecting on competing meanings, particularly when such reflection is engendered by a moment of perturbation.

Our second suggestion for a research line moving forward is also an extension, and one in the direction of upper-level mathematics (e.g., undergraduate mathematics and above). This suggestion is motivated by personal conversations with Anderson Norton, Shiv Smith Karunakaran, David Plaxco, and Paul Dawkins. These conversations revolved around the potential role of the figurative and operative distinction in the areas of proof and proving, logic, and the study of advanced mathematical objects. In short, those conversations have revolved around the tendencies of mathematicians and mathematics educators to reduce those areas to formalisms, including syntactical rules, proof structures or templates, logic models, algebraic systems, and other objects that they take to encapsulate operations. As Norton explained, such a framing emphasizes “the conceptualization of formal mathematics, as opposed to formalism deriving from the conceptualization of operations” (Personal communication). A productive line of research, which is suggested by Norton’s (2022) recently published book, may instead be to consider how the coordination of mental operations and meanings rooted in such provide a springboard for the formalizations required in upper-level mathematics.

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