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Research on students' and teachers' quantitative reasoning continues to underscore its importance for their learning and development. This importance requires that researchers continue to make strides in identifying salient and important ways of reasoning quantitatively. In this paper, we delineate four forms of quantitative reasoning to characterize both students' images of situations and their graphing meanings related to those images. Specifically, we differentiate between students conceiving quantities' changes via states reasoning, transformational reasoning, and gross or quantified covariational reasoning. We connect these forms of reasoning to their meanings for graphs when attempting to represent those quantities.

Keywords: Covariational Reasoning, Quantitative Reasoning, Graphing, Cognition

Quantitative reasoning involves conceiving a situation so that it entails measurable attributes (i.e., quantities) and relationships between them (i.e., quantitative relationships; (Smith III & Thompson, 2007; Thompson, 2011). *Covariational reasoning* involves conceiving the ways in which quantities vary in tandem (i.e., covary; Saldanha and Thompson (1998), Carlson et al. (2002), and Confrey and Smith (1995)). Together, quantitative and covariational reasoning (QCR) form a critical foundation for student development and teacher actions at all grade levels (e.g., Ellis, 2011; Johnson, 2015; Steffe & Olive, 2010; Tallman & Frank, 2020; Thompson, 1994). Over the past two decades, we have engaged in research to build models of students' and teachers' reasoning about experiential contexts in association with their QCR and graphing relationships in coordinate systems. In this report, we describe QCR forms that have emerged during this work: state, transformational, gross, and quantified. State and transformational reasoning entail thinking about quantities at distinct states. Gross and quantified reasoning additionally include images of covariation. We illustrate each form in the context of conceiving of a situation and constructing a graph to re-present quantitative relationships.

Covariation and Re-presentation

Carlson et al.'s (2002) and Thompson and Carlson's (2017) frameworks are two of the most used in the field, both because of the authors' prominence as QCR researchers and the generalized nature of the frameworks. We have drawn significantly on Carlson and Thompson's frameworks, with Carlson et al.'s (2002) attention to direction and amounts of change being a bedrock for our work in understanding students' graphing meanings (e.g., Liang & Moore, 2021; Moore, 2014; Paoletti et al., 2024). Thompson and Carlson's (2017) levels have aided us in describing student meanings for linear/non-linear relationships (Paoletti & Vishnubhotla, 2022).

A popular context-specific area for QCR research is students' coordinate-system and graphing meanings. Specific to our focus, coordinate systems provide a representational context for students to re-present quantitative relationships they have constructed in experiential contexts (e.g., taking a road trip or building a fenced pen). Lee et al. (2020) called this a *quantitative coordinate system*. Moore and Thompson (2015) termed such reasoning *emergent (graphical)* *shape thinking* (*EGST*) if such actions involve covariation. In our use of covariation frameworks to describe participants' EGST and quantitative coordinate systems, we experienced the need to clarify or reframe aspects of them. These needs included incorporating a more intentional focus on magnitude reasoning, non-variational forms of reasoning that maintain the spirit of EGST, and the extent to which amounts of change are coordinated and compared.

Four Forms of Reasoning: States, Transformations, and Gross/Quantified Covariation

In Table 1, we delineate four forms of reasoning to discuss QCR and EGST: (1) state reasoning; (2) transformational reasoning; (3) gross (covariational) reasoning; and (4) quantified (covariational) reasoning. Each form is quantitative; each involves two quantities existing in a paired, possibly deterministic relationship. Each involves understanding that each quantity can take on a multitude of magnitudes, and that changing the magnitude of one quantity might involve changing the magnitude of the other.

Table 1. Four Forms of Reasoning about Two (or more) Quantities

Reasoning	Description
State	Quantities' paired magnitudes are only dependent on the state under consideration. Changing from one pair to another involves imagining a different instantiation of the situation.
Transformational	Quantities' paired magnitudes are only dependent on the state under consideration. Changing from one pair to another involves transforming between instantiations and reconstructing the magnitudes.
Gross (Covariational)	Quantities' paired magnitudes are dependent on the state under consideration. Also, one pair can be changed to another by imagining a quantity's magnitude increasing/decreasing with increases/decreases in the other quantity's magnitude.
Quantified (Covariational)	Quantities' paired magnitudes are dependent on the state under consideration. Also, one pair can be changed to another by constraining gross covariation by an invariant property of their simultaneous changes.

A person engaged in *state reasoning* conceives the relevant quantities as occurring in distinct instantiations called states. The quantities' magnitudes can change in the sense that there can be different magnitudes at distinct states. The magnitudes at a state exist independent from those at other states. In the context of imagining a road trip, any moment in the trip is a state. At every state, there exists some distance from the start and some distance from the destination. Because each moment exists independent of another, conceiving another state involves switching attention from one state to the next state *and then* constructing and determining anew the two distance magnitudes (Figure 1a; the red magnitude represents the distance from the start and the blue magnitude represents distance from the destination). The shift in states produces distinct distance magnitudes, which may or may not have different sizes from other states. For example, a person engaged in state reasoning might focus on the halfway point in their journey and identify that the two distances are equal. They might then shift their focus to a rest stop and, with their attention shifted, conceive how far the stop is from the start and end. An individual engaged in state reasoning understands that the path constrains possible states. They also might determine the two distances sum to a constant magnitude at every moment in the trip, conceiving this as an invariant property defining the quantities' deterministic relationship at any state.

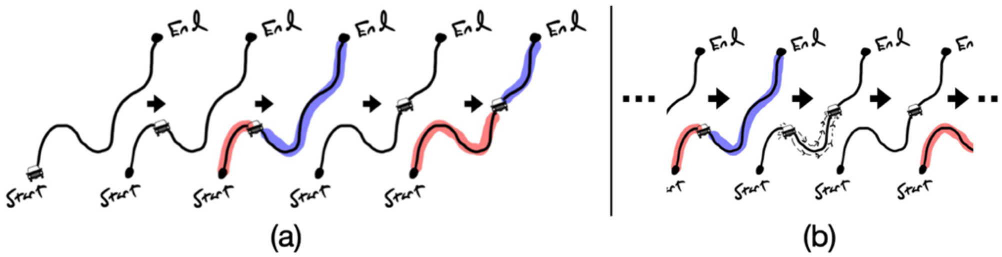

Figure 1. (a) State and (b) transformational reasoning about a road trip and distances.

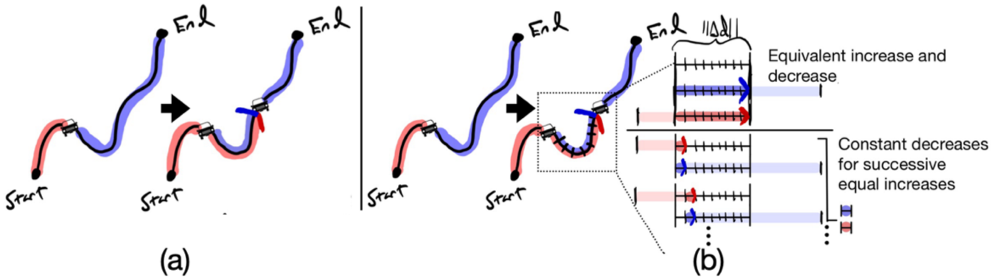

Figure 2. (a) Gross and (b) quantified reasoning about a road trip and two distances.

Transformational reasoning is equivalent to state reasoning, but the person imagines transforming the context from one state to produce another state. With a road trip, this involves imagining the car traveling along the road from one state to another, whether smoothly or in chunks (see Figure 1b, which is Figure 1a with the additional conception of the car traveling from one state to another). Like state reasoning, an image of the quantities' magnitudes must be constructed anew at the ending state due to the magnitudes not being sustained when conceiving the transformation. The states are no longer independent of each other, but the quantities' magnitudes constituting each state remain independent from those at other states. Both state and transformational reasoning allow for the quantities' magnitudes to change via being different sizes at different states. Each includes understanding the car's physical position can change, but neither includes images of the (co)variation of two quantities magnitudes.

A person engaged in *gross (covariational) reasoning* also understands quantities occur in distinct states and conceives states with respect to each other; any state is a *snapshot* occurring within the transformational image. Gross covariation additionally involves the quantities' magnitudes being sustained during a transformation. A person conceives one state of the quantities' magnitudes as dependent on the other state through a process of covariation; the ending state can be produced through a process of covariation emanating from the beginning state. In the case of the road trip, this might involve an individual reasoning that as they proceed from an early rest stop to some state later in the trip, their distance from the start increases and the distance they have remaining decreases (Figure 2a, indicated by the lengthening red magnitude and the shortening blue magnitude). They could conceive this as a loose process of simultaneous increase and decrease, or they could conceive the simultaneous increase and decrease in terms of the specific magnitude each quantity changes in total (e.g., explicitly identifying the magnitude increase in red and the magnitude decrease in blue). They might hold in mind that at any state during that covariation the two distances sum to a constant magnitude.

The aforementioned three forms of reasoning can entail holding in mind an invariant property defining the quantities' overall magnitudes at each state (e.g., they sum to a constant magnitude). A person engaged in *quantified (covariational) reasoning* also constructs a relationship in the *variations of each quantity* so that an invariant property *constrains their* *covariation* when transforming from one state to another. Gross reasoning dealt with general intervals of increase and decrease and possibly considering specified total change. Quantified reasoning involves more precisely comparing the quantities' variations. In the case of the road trip, this could involve a person reasoning that as they change states, *any increase in the distance* from the start necessitates a decrease of *equal magnitude* in remaining distance (Figure 2b). Or a person might reason that as the distance from the start increases by *successive and equal magnitudes*, the distance remaining decreases by *a constant magnitude* (Figure 2b). In the former, the person quantified the gross covariation by comparing and generalizing one distance's variation relative to the other (i.e., each quantity changes by the same magnitude). In the latter, the person quantified the gross covariation by comparing one quantity's variation across fixed variations in the other (e.g., one quantity's magnitude changes constantly for constant magnitude changes in the other). In each, the quantities' covariation, whether chunky or smooth, was refined through constructing and quantifying amounts of change. Reiterating a point made above, quantified reasoning need not entail reasoning about specified values; in the road trip situation an individual can coordinate the two distance magnitudes without determining specific values in a unit (Figure 2b). As with the previous three forms of reasoning, the person might also understand that the total distances sum to a constant magnitude at each state. They might also consider how the additional invariant property of covariation maintains the invariant sum property.

Four Forms of Reasoning: EGST

To illustrate these forms of reasoning in an additional context and in relation to EGST, we use a fencing task adapted by authors of this paper to generate models of students' thinking: *Imagine constructing a rectangular playpen out of 20 feet of fencing. Construct a graph representing all possible width and length combinations for the playpen.* The task prompts re-presenting the relationship between the dimensions as they occur in the playpen context.

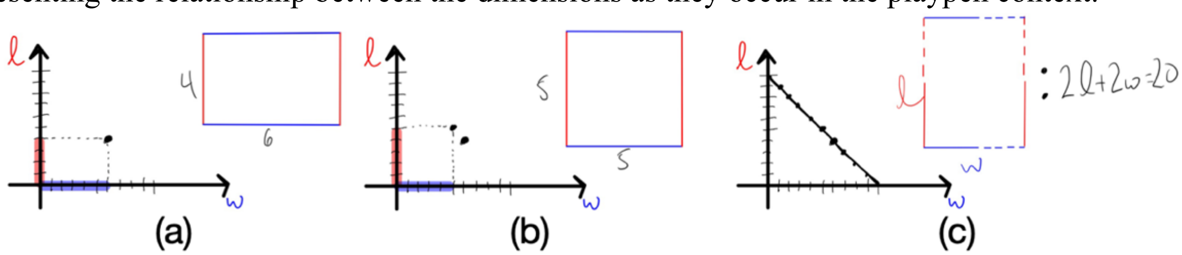

Figure 3. (a-b) Re-presenting paired magnitude states and (c) a sketched graph.

State reasoning involves understanding that each paired dimension is specific to a distinct, constructed playpen. The width and length magnitudes can change because of constructing a different playpen, with each state of paired dimensions being strictly dependent on the playpen one imagines constructing. The student might also understand the sum of playpen sides is 20 (e.g., $2w + 2l = 20$) at each state, and they might use this to determine dimensions at a state (e.g., choose $l = 5$ and determine w). Because the paired quantities are understood as dependent on a specific playpen, a student reasoning in this way produces a collection of points to capture each distinct playpen. They understand that each point captures a distinct playpen's paired magnitudes through the uniting of the axes-magnitudes re-presenting those quantities' magnitudes or values (Figure 3a-b). Having produced numerous points, they might also sketch a graph that captures a perceived visual pattern so that the graph captures every possible playpen (Figure 3c).

In the case of transformational reasoning, the person imagines transforming the pen from one state to the other; they have constructed a collection of playpens so that a playpen can be produced by modifying another. The potential transformations a student might imagine are numerous, creative, and unpredictable. One involves imagining squeezing the pen in one direction while stretching it in another. An explicit image of length and width magnitudes are not sustained during such a transformation. Re-presenting such a relationship graphically is equivalent to that of state reasoning, except the person might imagine how the point may move for each transformation of the pen. A drawn graph conveys the appropriate path for that movement. Stopping the movement along a drawn graph creates a state in which they can determine the width and length pairs for that playpen. An image of the quantities' magnitudes and their covariation is not sustained during the movement.

Moore and Thompson (2015) introduced EGST with an eye toward graphs as emerging through quantitative schemes constrained by an invariant property in the quantities' covariation. EGST is not immediately relevant to state or transformational reasoning due to an absence of images of (co)variation. We perceive both forms to be compatible with Moore and Thompson's more general purpose of articulating EGST as a quantitative meaning (Moore, 2021; Moore et al., 2022; Moore et al., 2024; Thompson, 2011, 2012, 2013). The collection of points emerges from an individual's reasoning about paired quantities' magnitudes, their re-presentation, and (potentially) an invariant property constraining the pairs (e.g., constant perimeter). We thus characterize such actions as *non-variational quantitative meanings* for graphing.

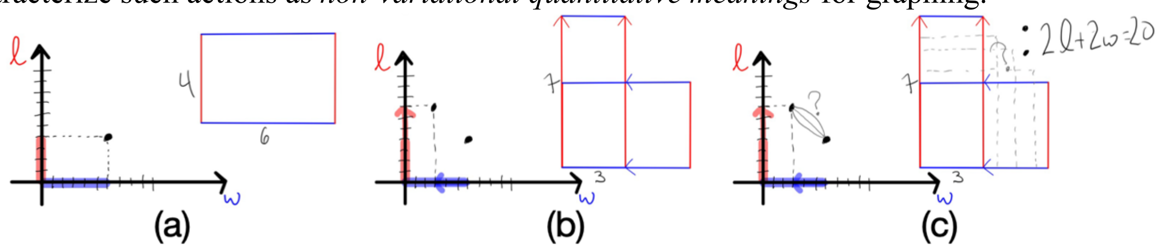

Figure 4. Re-presenting (a-b) paired magnitude states, (b-c) gross covariation, and (c) sketched graphs satisfying that gross covariation.

Gross (covariational) reasoning entails not only understanding that each pair of values corresponds to a constructed pen, but also that one pen can be conceived as a transformation from another pen through variations in the pen's length and width. Like above, a student reasoning in this way produces a collection of points to capture each state and they understand each point is a re-presentation of the state's paired quantities' magnitudes (Figure 4a-b). Differing from state or transformational reasoning, they also anticipate transforming from one point to the other via the quantities' covariation. For instance, they could reason about intervals of increase and decrease for the quantities that achieve the desired total variation amounts (Figure 4b-c). They could imagine these increases and decreases smoothly, and they might anticipate throughout the covariation that the quantities' total magnitudes at each state maintain the invariant property of a 20-foot perimeter. Because the quantities' covariation does not entail a quantified invariant property beyond (potentially) their resulting states maintaining the overall perimeter, the individual sketches a graph that results in that overall covariation. In doing so, they may or may not anticipate that the graph's emergence can occur in multiple ways to re-present that amount of variation between states (e.g., three possible graphs are shown in Figure 4c). Furthermore, the individual may or may not anticipate that, for whichever graph is accurate, the pen will transform in whatever way produces that graph (see dotted lines in Figure 4c representing an anticipated pen transformation). As with the previous forms of reasoning, having produced numerous points, a student might anticipate the sketch of a graph that captures a perceived pattern, the gross covariation, and a point for every state.

Both gross and quantified (covariational) reasoning involve determining invariant properties of covariation that constrain the transformation of the pens. With gross reasoning, the quantities' covariation within the two states is only constrained by achieving a total amount of variation. With quantified reasoning, an invariant property between the quantities' covariation is constructed so that one state can be transformed to another state in a way that additionally satisfies some property further quantifying the directional covariation. A student reasoning in this way might first produce one point re-presenting the pen's dimensions at some chosen state (Figure 4a from above). Then, they might use gross covariation and a conceived invariant property of that covariation to capture additional points or sketch a graph via re-presenting that property. An example of this is reasoning that the width and height magnitudes must covary such that any increase or decrease in width requires an equivalent decrease or increase in length, respectively (Figure 5a). For example, a width increase of 1 requires a height decrease of 1. The person's graph would then emerge as a result of coordinating the quantities' covariation under the constraints of that invariant property (Figure 5b). As with the road trip situation, a person could also reason that for successive, equal decreases in height, the width increases by a constant amount. A graphing action reflecting this might be the student denoting equal decreases in the height-axis magnitude and pairing that with equal increases in the width-axis magnitude. In either case, the individual might also consider how the invariant property in the quantities' covariation results in states that satisfy the invariant property of a constant perimeter.

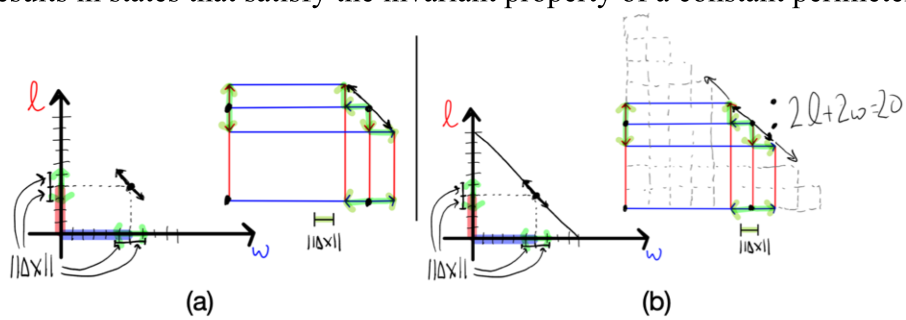

Figure 5. Re-presenting (a) quantified covariation and (b) a sketched graph constrained by that covariation.

All forms enable sketching a graph that captures states precisely. Using gross or quantified reasoning, an individual can construct a graph via EGST by first constructing points to capture snapshots and then sketching a graph that captures those states and invariant properties of covariation. Gross reasoning does not entail relative comparisons of variations, whereas quantified entails comparing variations between quantities (e.g., both quantities increase by the same amount) or within quantities (e.g., for equal increases in one quantity, the other quantity increases by decreasing amounts). With quantified reasoning, if the invariant property is quantified in terms of generalized properties (e.g., for a unit-magnitude change in quantity A, quantity B increases by 10% of its prior magnitude), then an individual can construct a graph via EGST strictly using one state and generating *all* other states using that invariant property.

A Brief Return to Informing Frameworks

We distinguish certain forms of non-variational reasoning from forms of QCR, and there are numerous connections to the frameworks within which we experienced the need to clarify these forms (Carlson et al., 2002; Thompson & Carlson, 2017). One need was introducing forms of reasoning that foreground magnitude reasoning. Whereas prior frameworks do not make magnitude reasoning explicit and often refer to or use numerical value, we describe our forms of reasoning with respect to magnitudes to be consistent with Thompson's (2014) emphasis on magnitude-based QR. As a second need, we clarify distinctions between Carlson et al.'s (2002) directional and amounts of change reasoning. Gross reasoning can entail loose intervals of increase and decrease as well as identifying specified amounts of increase or decrease. The former is consistent with directional reasoning and the latter is consistent with amounts of change reasoning. Quantified reasoning is also consistent with amounts of change reasoning, but it makes explicit the coordinated comparison of amounts of change. To illustrate, gross reasoning includes understanding that as quantity A increases from 1 to 2 to 3, quantity B increases from 2 to 5 to 9. Quantified reasoning adds operations to compare these increases and conceive quantity B increasing by *increasing amounts* as quantity A increases by *successive constant amounts*. Thus, gross and quantified reasoning underscore important differences in how students can construct and reason about amounts of change. As a third motivating need for our forms of reasoning, Thompson and Carlson's (2017) framework foregrounds distinctions between images of variation, and specifically between chunky and smooth images (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013). Our forms differentiate between reasoning that entails thinking of quantities as merely taking on different amounts (i.e., state and transformational reasoning) and that which entails explicit images of quantities' covariation (i.e., gross and quantified reasoning). Differences between smooth and chunky images are critical and relevant to each form of reasoning we identify, and we thus perceive our forms to complement those distinctions.

Closing

A question remains as to how an individual might conceive a drawn graph that connects the points they have produced through state or transformational reasoning. With both forms of reasoning, a student might conceive the drawn graph as capturing the infinite number of states that exist and maintain an invariant property among those states (e.g., constant fence amount and rectangular dimensions). In the case that a student has previously constructed and abstracted EGST meanings, the production of a graph via state reasoning can provide the source material for their constructing the quantities' covariation. For example, a phenomenon could be so physically and quantitatively complex that it is difficult for an individual to construct a covariational relationship within the phenomenon. They may instead determine paired values, construct an invariant property between the paired values or produce a substantial collection of them, and then sketch a graph. With the graph at hand, they could then engage in EGST to infer the covariational relationship between the quantities in the phenomenon. If compelled, they could then return to the phenomenon to judge the viability and further explore the relationship.

Having already constructed and abstracted EGST meanings, a student reasoning as above anticipates that in sketching their graph they are producing a trace constrained by covariational properties. What of a student who has not constructed and abstracted EGST meanings, but produces a drawn, continuous graph using state or transformational reasoning? Based on our work and the work of our colleagues (Byerley & Thompson, 2017; Moore, 2021; Moore et al., 2022; Moore, Silverman, et al., 2019; Moore, Stevens, et al., 2019; Thompson, 2013, 2016; Thompson et al., 2017), we are confident that without an explicit and persistent subsequent focus on quantities and their covariation, students are more likely to develop meanings rooted in indexical or declarative associations with mathematical facts (e.g., a line rising up-and-to-the-right means positive slope) than meanings stemming from QCR that can accommodate novel and unconventional situations. Without an explicit and persistent focus on quantities and their covariation, students are placed in a situation where they can associate the shape of a graph with capturing distinct paired values, invariant properties among those total values, and declarative facts, but the graph's emergence as an organic phenomenon between states is left unaddressed.

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