Competing Meanings, Perturbation, and Engendering Shifts in (Prospective) Teacher Meanings

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Competing meanings, perturbation, and engendering shifts in (prospective) teacher meanings

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Through building models of student thinking, researchers have identified quantitative reasoning as a foundation for students' mathematical development. This has generated a need to support teachers' capacity to teach for such reasoning. In this paper, we discuss a perspective on supporting prospective and practicing teachers in constructing meanings that foreground quantitative reasoning and research-based models of student thinking. We call this perspective competing meanings and describe it as a three-phase cognitive process. First is a perturbation of an extant meaning resulting from its enactment. Second is the construction of an alternative meaning to reconcile the perturbation. Third are acts of reflection to compare extant and alternative meanings. We introduce the competing meanings perspective and illustrate it with examples. We also discuss theoretical resources that inform this perspective, including Piagetian constructs, quantitative reasoning, and mathematical knowledge for teaching.

KEYWORDS
cognition, learning theory, quantitative reasoning, teacher education, teacher knowledge

Introduction

Quantitative reasoning refers to the ways in which individuals conceive of and reason with measurable attributes constituting contexts (Smith and Thompson, 2007; Thompson, 2011; Thompson and Carlson, 2017). In the context of a growing number of topics including fractional, proportional, algebraic, and function reasoning, researchers have developed models of student thinking that identify quantitative reasoning as a critical foundation for mathematical development (e.g., Karagöz Akar et al., 2022; Steffe and Olive, 2010; Thompson and Carlson, 2017). These researchers' findings also imply that factors influencing students' educational experiences do not sufficiently support students' quantitative reasoning. Whether through improving curricular materials, engendering teacher knowledge development, or supporting teachers in enacting productive teaching practices, a pressing need is to understand how to aid teachers in cultivating their students' quantitative reasoning.

Our team has responded to this need by enacting a research program to understand and support not only students' quantitative reasoning, but also that of prospective and practicing teachers.[1] We have unsurprisingly found that supporting teachers' quantitative reasoning is complex. Because extant curricula and pedagogical practices often do not support quantitative reasoning, teachers' meanings for key concepts vary widely in the extent they entail quantitative reasoning. Supporting teachers' quantitative reasoning thus requires an investigation of the interplay between teachers' extant mathematical meanings and quantitative reasoning. One part of this investigation is exploring the extent to which their current meanings entail quantitative reasoning. A second part of this investigation is exploring how to engender teachers' quantitative reasoning so that it might lead to productive evolutions in their mathematical meanings. A third part of this investigation is understanding the foundational impact of these evolutions on teachers' *mathematical knowledge for teaching* (*MKT*).

In our investigations, we have frequently experienced cases in which teachers' extant meanings are disconnected from or incompatible with quantitative reasoning. Their extant meanings can also act as roadblocks to their constructing quantitative reasoning-based meanings, and transforming those meanings into MKT. We have worked to understand this phenomenon, including how teachers might reconcile perturbations of extant meanings to construct and come to privilege alternative, quantitative reasoning-based meanings when their extant meanings initially impede such a result. In this paper, we present the *competing meanings* perspective that has emerged from this work.

In what follows, we first provide background theory that informs the competing meanings perspective. The background theory includes concepts from Piaget (2001), von Glasersfeld (1995), Thompson's (2013, 2016) theory of meaning, and Harel's (2013) intellectual need. We then integrate these informing theories to introduce the competing meanings perspective. Our goal in introducing the competing meanings perspective is to explicate a form of meanings development that entails (a) provoking and problematizing extant meanings, (b) supporting the construction of quantitative reasoning-based meanings, and (c) prompting interactions between the two. The competing meanings perspective uses this progression to bridge its informing theories with Silverman and Thompson's (2008) theory of MKT. Namely, the competing meanings perspective identifies how teachers' meanings might shift from those meanings they currently hold to meanings researchers have built based on models of students' mathematical realities. The construction of meanings that better reflect students' mathematical realities is a foundational step in supporting the development of MKT. The development of knowledge that has pedagogical power (i.e., MKT) requires meanings which reflect, honor, and build upon students' mathematics (Johnson et al., 2022; Liang, 2025; Silverman and Thompson, 2008; Tallman and Frank, 2020; Tallman et al., 2024).

Although this paper is primarily an introduction of the competing meanings perspective, the competing meanings perspective emerged from decades of empirical work with middle grades, secondary, and undergraduate students and teachers. We draw on this work to provide brief illustrations related to the competing meanings perspective through an example task sequence. Relatedly, we identify task-design principles and considerations that embody the competing meanings perspective. Still, this paper primarily presents a theoretical framework and illustrative applications, with additional empirical work currently underway. We thus close with potential implications and future work by drawing specific attention to areas of theory left to flesh out or connect with.

Informing theory and background

Our goal is to present a theory that articulates how teachers' meanings might shift from their extant meanings to those that are more productive for supporting their students' mathematical development, particularly when the former are in some ways incompatible with the latter. Piaget's (2001) genetic epistemology and von Glasersfeld's (1995) extension of Piaget's work provide our meaning-focused work a general orientation. We adopt Thompson's theory of meaning (Thompson, 2016; Thompson et al., 2014) and quantitative reasoning (Smith and Thompson, 2007; Thompson, 2011; Thompson and Carlson, 2017) to enable us to specify tangible meanings. In order to articulate a form of learning specific to the competing meanings perspective, we also use Harel's (2013) notion of intellectual need. Lastly, in order to support using the competing meanings perspective to bridge the aforementioned theories and the knowledge relevant to teaching, we situate these theories within relevant constructivist theories of MKT (Liang, 2025; Silverman and Thompson, 2008; Simon, 2006; Tallman and Frank, 2020). As an aid for the reader, we provide a glossary of important terms in the Appendix.

Piaget's genetic epistemology

We adopt Piaget's genetic epistemology as an orienting lens for explaining cognitive activity. Focused on children's cognitive development, Piaget generated constructs to characterize knowledge states, differentiate between types of knowledge, and describe knowledge development. His constructs of *assimilation*, *perturbation,* and *accommodation* are most relevant to the present paper and we point the reader to Dawkins et al. (2024) for an extensive collection of Piaget's theory as used by mathematics education researchers.

Assimilation is the process by which an individual conceives a present experience via their current conceptual structures (von Glasersfeld, 1995). In some cases, assimilation results in an unexpected experience, which engenders a state of *perturbation* (von Glasersfeld, 1995). A perturbation can occur for several reasons. For one, an individual might obtain an unexpected result after enacting a conceptual structure, thus yielding a sense of perplexity. Or the learner might experience an obstacle in carrying out their conceptual structure. In each case, the learner experiences a mismatch between what their current conceptual structures lead them to anticipate and what they experience.

Having experienced a perturbation, an individual engages in activity to reconcile that cognitive state. One form of reconciling a perturbation involves affective and coping responses, such as anxiety leading to disengagement (Tallman and Uscanga, 2020). Another form of reconciliation is that of *accommodation*. Accommodation is an act of learning via the elimination of a perturbation through a cognitive construction or reorganization. One form of accommodation is modifying the conceptual structure used in assimilation to address the perturbation. Another form of accommodation is constructing a novel

1 From here forward we use *teachers* to include both prospective and practicing teachers. We include the terms *prospective* or *practicing* if it is necessary to identify a particular population.

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conceptual structure. A third form of accommodation is enacting an alternative extant conceptual structure.

Thompson's theories of meaning and quantitative reasoning

We have to this point left the term *conceptual structure* ill-defined. Thompson introduced the intertwined theories of meaning (Thompson, 2013, 2016; Thompson et al., 2014) and quantitative reasoning (Smith and Thompson, 2007; Thompson, 1993, 2011) to provide an operational definition of conceptual structures. Thompson's theory of meaning is rooted in Piaget's notion of a scheme, or "an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity" (Thompson et al., 2014, p. 10). Harel and Thompson built on the notion of a scheme and formalized the term *meaning* to refer to "the space of implications that the current understanding mobilizes—actions or schemes that the current understanding implies, that the current understanding brings to mind" (Thompson et al., 2014, p. 12). Their idea of a meaning underscores that while a scheme is used in assimilation, the enactment of a scheme brings forth a space of implications that includes other schemes. This space is collectively referred to as a meaning.

Harel and Thompson's notion of meaning is critical to our focus due to its implications for teaching. It is not sufficient for a teacher to only understand an idea via a single scheme; a teacher should develop an interconnected system of schemes for understanding a mathematical idea. This system of schemes is a meaning, with assimilation to any scheme in the system implying the other schemes that constitute the meaning. Similarly, while it is important that a teacher is able to understand a student's thinking via assimilating a student's actions to a scheme, it is just as critical that they also consider the implications of that student's thinking relative to other mathematical ideas and instructional goals (Johnson et al., 2022; Liang, 2021, 2025; Teuscher et al., 2016). A teacher assimilating a student's thinking should bring forth a sophisticated system of implications and relationships with other schemes and meanings that the teacher can act upon in interaction with the student (Liang, 2021, 2025).

We draw on Thompson's (1993, 2011), and Thompson et al. (2014) theory of quantitative reasoning to discuss a tangible set of meanings and schemes. *Quantitative reasoning* refers to meanings that involve conceiving situations in terms of measurable attributes (i.e., quantities) and relationships between those attributes (i.e., quantitative relationships). To illustrate, the object of a graph can be conceived as having two measurable attributes, which in the case of the Cartesian coordinate system are two oriented distances conventionally represented along orthogonal axes; these two measurable attributes can be conceived in terms of relationships that define the graph so that the graph is understood as emerging as a consequence of those two quantities (Figure 1). Moore (2021) and Moore and Thompson (2015) named this quantitative meaning for a graph as *emergent (graphical) shape thinking* (EGST) to underscore that although instruction

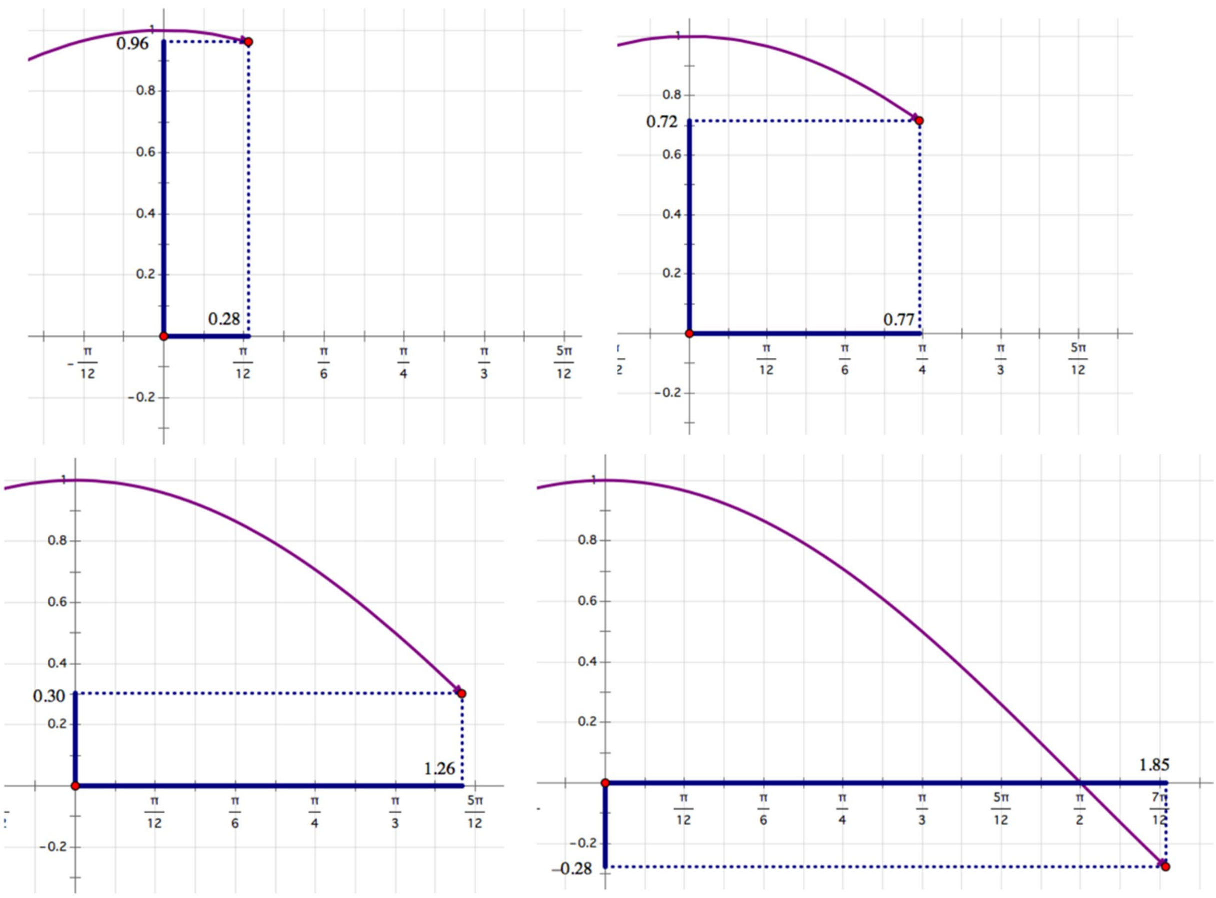

FIGURE 1
Instantiations of emergent (graphical) shape thinking (Moore, 2021, p. 156).

frequently focuses on visual properties of shape, productive meanings for graphs entail understanding how a graph results from quantities changing in tandem.

Covar*iational reasoning* is the form of quantitative reasoning that involves how quantities vary in tandem (Carlson et al., 2002; Saldanha and Thompson, 1998; Thompson and Carlson, 2017). A growing number of researchers have identified important nuances in student and teacher thinking in this area (e.g., Ellis et al., 2020; Ferrari-Escolá et al., 2016; Johnson, 2015a,b; Patterson and McGraw, 2018; Yu, 2024). Most relevant to our work, Carlson et al. (2002) identified a covariation-based meaning that involves *directional* and *amounts of change* reasoning. Ellis et al. (2015) illustrated such reasoning as productive for supporting students' meanings for exponential relationships in the context of (magic) plant growth and the quantities height and time (Figure 2). The students' meanings involved their constructing the directional covariation of quantities via reasoning that as *time increases*, *height increases*. The students' meanings also involved their constructing the amounts of change covariation of quantities via reasoning as *time increases additively*, *height increases by increasing amounts*. Their meaning further included coordinating *additive changes in one quantity* with *multiplicative changes in the other* so that the latter preserved a constant ratio for a constant time period.

There are several reasons we situate our work within quantitative and covariational reasoning beyond their relationship with Piaget's epistemology. For one, researchers have indicated that students' quantitative reasoning provides a flexible and powerful foundation for their development of STEM ideas inside and outside the classroom (e.g., Carlson et al., 2022; Ellis et al., 2015; Fonger et al., 2020; Jones, 2022; Jones and Ely, 2023; Karagöz Akar et al., 2022; Lee et al., 2020; Moore et al., 2022; Paoletti and Vishnubhotla, 2022; Yoon et al., 2021). For another, researchers' empirical work in this area has provided models of students' mathematical realities that spotlight their constructing powerful mathematical meanings. Because they are rooted in students' mathematical realities, as opposed to the mathematics of adults or experts, these models provide researchers and teachers tools to design learning environments that privilege students' realities (Steffe and Thompson, 2000). In fact, Tallman and Uscanga (2020) illustrated that learning opportunities foregrounding students' quantitative and covariational reasoning are critical to addressing mathematics anxiety.

Specific to our work, we leverage models of students' quantitative and covariational reasoning to investigate teachers' meanings with sensitivity to the mathematics of students. Drawing on this literature base enables moving beyond a focus on teachers' extant meanings and traditional curricular approaches to account for meanings more productive and relevant to students' mathematical realities. Responding to recent calls for reshaping mathematics education to better reflect the learner's perspective (Adiredja, 2019; Hackenberg et al., 2024; Steffe et al., 2014; Thompson, 2013), we aspire to support teachers in learning and teaching for a research-based mathematics of students.

Intellectual need

An individual in a state of perturbation experiences an accompanying need to reconcile that perturbation. We draw on Harel's (2008, 2013) notion of *intellectual need* to clarify the perturbations relevant to our perspective. Harel defined intellectual need as "a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, his or her current knowledge" (2013, p. 122). Importantly, Harel's intellectual need refers to states of perturbation that afford learning. A state of confusion is not a sufficient condition to occasion a state of intellectual need. Rather, a researcher is positioned to claim an individual has experienced an intellectual need when they experience a perturbation *and* the individual has the capacity to construct the meanings needed to reconcile that perturbation (Harel, 2013; Paoletti et al., 2024; Weinberg et al., 2023).

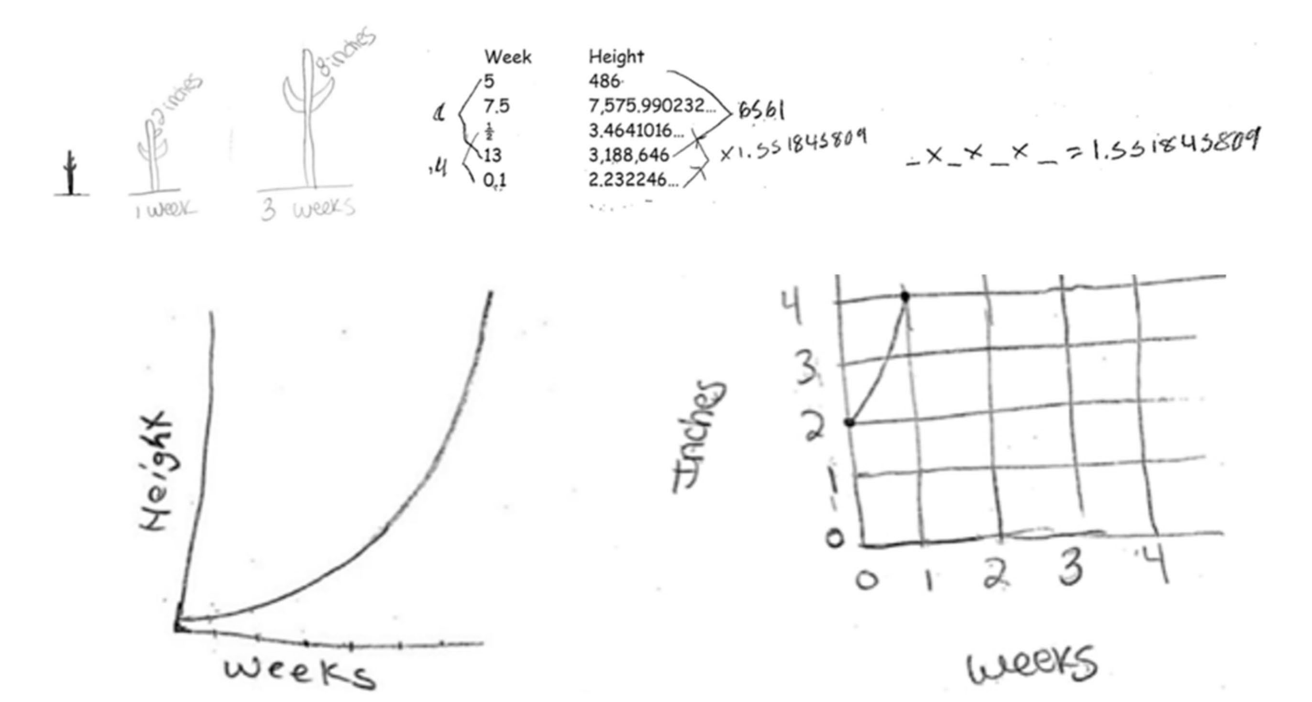

FIGURE 2
Students' coordinating height and time (Ellis et al., 2015, pp. 143, 147, and 149).

Intellectual need is an important construct because it orients us toward not only seeking to engender a perturbation, but also having in mind the ways in which that perturbation will act as a catalyst for constructing alternative meanings to reconcile it. In detailing his notion of intellectual need, Harel (2008) drew explicit attention to the difference between resolving a perturbation with or without awareness of how a meaning reconciles a perturbation. He used *epistemological justification* to refer to the knowledge constructed when an individual becomes explicitly aware of how a constructed meaning reconciles a perturbation. Intellectual need thus orients us to consider how an individual reflectively compares perturbation-creating meanings to those meanings that reconcile perturbations.

Mathematical knowledge for teaching

The awareness Harel (2008) captured with epistemological justification is critical to have in mind when working with teachers. Such reflective activity is critical to a teacher developing a knowledge base they can act upon in their practice. A teacher's professional knowledge base is formed not only by the meanings they can bring forth when solving a mathematical problem, but also by their capacity to make a personal meaning an explicit object of reflection outside of its immediate use (Ball and Bass, 2000; Hill et al., 2008; Liang, 2025; Silverman and Thompson, 2008; Tallman and Frank, 2020; Tallman et al., 2024; Thompson, 2016). Discussing the foundational teacher knowledge work of Shulman (1986) and Dewey (1902), Tallman et al. (2024) summarized that reflecting on one's own thinking leads to the psychologization of subject matter and "is the essential process by which content knowledge is endowed with pedagogical utility" (p. 6). Liang (2025) further illustrated that such a process is critical for teacher understanding and comparing their personal meanings to those they attribute to their students.

We draw on Silverman and Thompson's (2008) perspective on MKT in order to frame a teacher's professional knowledge base.[2] Silverman and Thompson described MKT as a developmental process that involves the construction of personalized knowledge that is, over time, transformed to have pedagogical power. Silverman and Thompson (2008) summarized,

> A teacher has developed knowledge that supports conceptual teaching of a particular mathematical topic when he or she (1) has developed a [*key developmental understanding*] within which that topic exists, (2) has constructed models of the variety of ways students may understand the content (decentering); (3) has an image of how someone else might come to think of the mathematical idea in a similar way; (4) has an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea; (5) has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas. (p. 508).

2 We note that both Tallman and Liang have elaborated on Silverman and Thompson's (2008) MKT perspective, including situating it with respect to other MKT perspectives. Additionally, Thompson (2016) has since referred to his perspective on teacher knowledge as mathematical meanings for teaching.

Simon's (2006) notion of a *key developmental understanding* (KDU) underpins Silverman and Thompson's (2008) MKT. Simon introduced KDUs under the premise that a teacher's mathematical meanings influence their pedagogical choices and actions. A KDU identifies a meaning that is pivotal in students' mathematical development and is thus important for a teacher to hold. KDUs connect ideas, provide foundations for learning ideas, and support solving novel problems. KDUs are immediately beneficial to solving a problem; they are also generative via supporting future learning and connecting what appear to be different situations. Because of their generativity and flexibility, Simon argued that mathematics education researchers need to identify KDUs and identify how to design instruction to target them, particularly with teachers.

KDUs are effortful in their construction (Simon, 2006). They typically develop over multiple experiences reasoning in compatible ways and through reflecting on that reasoning. He explained that the construction of a KDU is unlikely to be the result of teacher explanation or demonstration, or the result of a single engagement in a task. The construction of a KDU also involves a conceptual advance (e.g., an accommodation), so that after an individual constructs a KDU they conceive relevant situations in fundamentally different ways than prior to its construction. These differences are nontrivial.

Returning to Silverman and Thompson's (2008) MKT, personal meanings form the foundation for a teacher's actions. A teacher's meanings vary in the extent they enable constructing a KDU, understanding student thinking, or supporting student learning. In order for a teacher's meanings to develop into productive MKT, the teacher must transform them in ways sensitive to students' mathematical realities (Silverman and Thompson, 2008; Thompson, 2016). A critical step in a teacher constructing MKT is thus constructing meanings that afford understanding and honoring students' mathematical realities. Such meanings provide a critical foundation for a teacher because their *potential* for pedagogical power is significant. Meanings that enable understanding and honoring students' mathematical realities provide teachers a foundational knowledge base through which they can better discern and act upon student reasoning (Baş-Ader and Carlson, 2022; Carlson et al., 2024; Ellis, 2022; Hackenberg et al., 2024; Liang, 2025; Teuscher et al., 2016). Repeating a point from above, this is a reason we situate our competing meanings perspective in the quantitative reasoning literature base. That literature base provides models of students' mathematical realities from which we identify KDUs that can act as foundational parts of teachers' meanings.

Before we continue, we acknowledge that personal meanings and KDUs alone are not sufficient to act pedagogically in ways that honor students' realities. A phase of MKT development involves envisioning instructional settings and interactions that might afford students constructing a KDU using their current ways of operating. MKT development further involves a teacher understanding how support a student in using a constructed KDU to learn other mathematical ideas. Silverman and Thompson (2008) described that such transitions move a KDU from "having pedagogical *potential* to an understanding that does have pedagogical power" (p. 502). Tallman et al. (2024) recently provided an empirical account of a teacher constructing and transforming meanings to have pedagogical power in the context of constant rate of change. Our focus in introducing the competing meanings perspective is on the construction of personal meanings and KDUs, which act as a foundation for MKT. We return to raise unaddressed aspects in the closing of the paper and provide suggestions for future work in this area.

The competing meanings perspective

Integrating the ideas described in the prior section, we introduce the competing meanings perspective in two parts. First, we describe competing meanings in terms of the three-phase sequence of cognitive processes we intend to capture. Second, we situate these processes in different grain sizes of intellectual need, highlighting how teachers may experience and reconcile perturbations at the level of tasks, meanings, and the broader implications of meanings, which contributes to the development of a KDU.

The processes of the competing meanings perspective

Competing meanings captures cases in which a learner experiences a perturbation after having enacted an extant meaning; enacts an alternative meaning to reconcile that perturbation; and reflectively compares the extant meaning and the alternative meaning (Figure 3). The first process of the competing meanings perspective involves the problematization of an extant meaning via an act of assimilation that engenders a perturbation. The second process of the competing meanings perspective involves a learner reconciling their perturbation. They do this by constructing an alternative meaning. This alternative meaning may require the construction of novel schemes, or it could involve enacting available schemes in a novel way. For instance, in attempting to understand a novel graph, the learner might not have reasoned about covarying quantities' amounts of change before, thus requiring them to construct and coordinate quantities' amounts of change. Alternatively, the learner might have reasoned about covarying quantities' amounts of change in other contexts but not in a present task context. They might then reorganize their amounts of change reasoning so that it accommodates the present task context.

Critical to the competing meanings perspective is a third process that involves comparing both their extant and alternative meanings. The individual is perturbed as to why their extant meaning results in a perturbation whereas the alternative meaning reconciles that perturbation; the learner is perturbed by the disparate nature of their meanings. By disparate, we mean that, *in that moment*, the learner infers their two meanings entail important differences due to the meanings resulting in different states of understanding upon enactment. This perplexity positions the learner to take each meaning as an object of thought and critically compare them—hence the name competing meanings—to reconcile that perturbation.

Through engaging in the reflective practice of comparing meanings, the learner can make aspects of each meaning explicit. They also can compare those aspects in terms of their affordances and constraints for the present task, thus constructing an awareness of how particular aspects of each meaning either lead to a perturbation or enable solving the task to their satisfaction. This collective process results in an accommodation in the form of their constructing an epistemological justification for their initial perturbation and subsequent reconciliation.

Having constructed an epistemological justification for their initial perturbation and subsequent reconciliation, the learner can compare their meanings at an additional level. The learner can become curious about broader implications of their meanings. They can ask, "I wonder how this alternative meaning relates to other ideas and problems?" To reconcile this uncertainty, the learner might attempt to enact and reflect upon both their extant and alternative meanings in the context of tasks or topics not yet considered. For instance, if the topic was slope or rate of change, the learner might consider the implications of extant and alternative meanings relative to learning derivatives, slope fields, or phase planes. Or they might consider the implications of different meanings for understanding rate of change in different coordinate systems. Regardless of topic, this additional level of comparison affords the learner repeated opportunities to enact each meaning, make explicit aspects of each meaning, and develop an understanding of the relative affordances and constraints of each meaning.

Repeated processes of perturbation and intellectual need

Intellectual need informs our description of the above cognitive processes and perturbations in several ways. With respect to the initial perturbation, a researcher is positioned to claim the learner experiences an intellectual need at the *level of the present task*. There is an intellectual need for an alternative meaning whose enactment satisfactorily accomplishes the conceived task. With respect to the second perturbation, a researcher is positioned to claim that the learner experiences an intellectual need at the *level of meanings*. There is an intellectual need for analyzing the meanings with respect to each other and against the backdrop of the tasks they have engaged in. Through comparing the

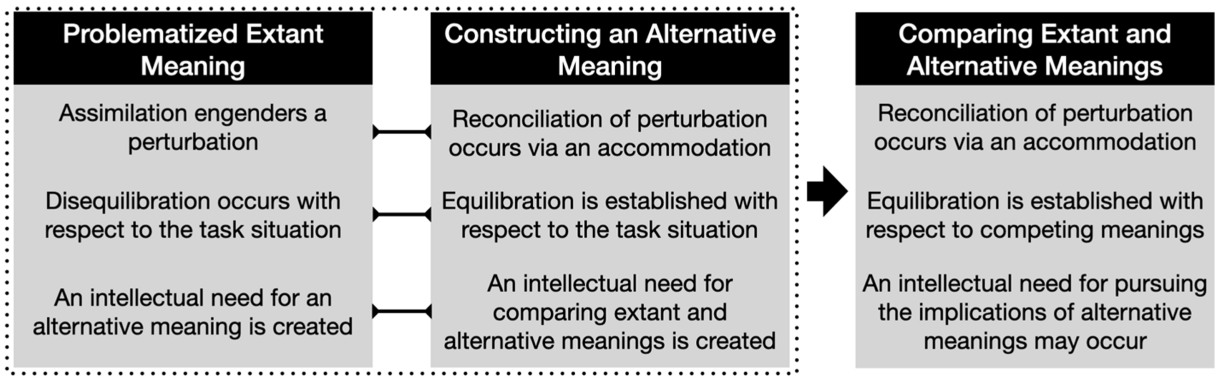

FIGURE 3
The competing meanings perspective.

meanings, the learner satisfies this intellectual need by identifying disparate features of their meanings. They become aware of how the meanings relate to and are distinct from each other, thus constructing an epistemological justification with respect to the meanings in relationship to experienced tasks. We note that the outcome of such reconciliation can vary. The learner may come to perceive one meaning as a consequence or derivative of the other. The learner may come to perceive one meaning as more efficient than the other. The learner may come to perceive one meaning as more flexible than the other. The learner may come to perceive one meaning as more relevant to their teaching than the other. The learner may come to perceive one meaning as superior to the other for a variety of reasons. These are but a few potential outcomes.

With respect to the last perturbation referenced in the previous section, a researcher is positioned to claim that the learner experiences an intellectual need at the *level of the implications of their meanings*. There is an intellectual need for understanding each meaning in the broader landscape of mathematical development and education. As the learner explores these implications with respect to the present topic, topics related to and building upon the present topic, or other educational resources like curricula and standards, they satisfy this intellectual need by constructing a system of implications of the meanings. We find it important to mention that this process can cause additional reflection on how the extant and alternative meaning relate to the initial task. By pursuing the implications of each meaning, the learner further develops each meaning. This ongoing development can shape the learner's appraisal of the meanings with respect to the initial task or other previous tasks. For instance, as they explore the alternative meaning's relevance to subsequent topics, they may come to reconceive those topics in ways that become personally powerful when compared to their previous understandings. This can result in the learner increasing the extent to which they privilege a meaning in broader contexts or topics.

The multi-level aspect of the competing meanings perspective is driven by the idea of a KDU. Recall that KDUs are meanings pivotal to mathematical development and form a foundation to the construction of MKT. KDUs are immediately beneficial to solving a problem. KDUs also connect various ideas, provide foundations for learning other ideas, and support solving novel but related problems (Silverman and Thompson, 2008; Simon, 2006). A researcher or teacher-educator cannot expect a teacher to construct and privilege an alternative meaning based solely on its initial enactment and reconciliation of a perturbation. Thus, in creating the competing meanings perspective, we incorporate an intentional focus on the individual reflecting on their meanings at multiple grain sizes. The individual reflects on their meanings with respect to the present task. The individual reflects on their meanings with respect to their implications not only for other tasks, but also for other topics and ideas. It is during these reflective processes at various grain sizes that the individual can identify the extent a particular meaning is *key* to development and worthwhile to pursue in teaching.

Illustrating competing meanings

Consider a hypothetical teacher, who we name Blinder. For a particular concept, Blinder has constructed a meaning we denote by M_a, which has served as productive throughout his schooling and teaching experience. Entering our professional development, we might desire that Blinder construct an alternative meaning that better reflects research-based models of students' mathematics. We denote this alternative meaning by M_b. In working with Blinder, we determine that he holds meaning M_a and that meaning M_a and M_b are disparate; M_a is not a foundational way of thinking for M_b and, in this case, will inhibit Blinder's construction of M_b and his coming to view it as relevant to his teaching and students' learning. This raises the question: how do we engage with Blinder in a way that honors M_a while also affording his constructing M_b so that it becomes relevant to his teaching? This is a situation that motivates our competing meanings perspective.

We use the topics of linear relationships, the sine relationship, and, more broadly, the function concept to provide concrete illustrations of our perspective. The meanings we discuss for each topic are informed by the same KDU. Recall that a KDU connects various ideas and provides the foundation for learning numerous topics because it is a *key* form of reasoning that enables approaching each topic in a compatible way. We thus first introduce the KDU driving our work, situate it with respect to specific meanings for each topic, and identify a disparate extant meaning.

A KDU for function and relationships

Much of the research on students' quantitative reasoning—particularly that on covariational reasoning—is directly or indirectly related to the function concept and their representations (Carlson et al., 2002; Carlson and Oehrtman, 2004; Oehrtman et al., 2008; Thompson, 1994; Thompson and Carlson, 2017). This collective body of work illustrates that a KDU for function involves a relationship meaning that centers the simultaneous variation of two or more quantities' values (Carlson, 1998; Carlson and Oehrtman, 2004; Ellis, 2011; Moore, 2025; Moore and Paoletti, 2015; Oehrtman et al., 2008; Paoletti, 2020; Paoletti and Moore, 2018; Paoletti et al., 2024; Patterson and McGraw, 2018; Saldanha and Thompson, 1998; Thompson, 1994; Thompson and Carlson, 2017). Such a meaning backgrounds notions of dependency or causation, instead foregrounding quantities as simultaneously occurring with their simultaneous variation constrained by mathematical properties of covariation (Saldanha and Thompson, 1998).

A meaning comprised by the simultaneous variation of two quantities provides a powerful foundation for several concepts including functions and their graphs (Moore et al., 2022; Paoletti, 2020; Paoletti et al., 2024). This meaning involves understanding that a function and its inverse are born from and hence capture the same invariant relationship. Allowing for multi-valued functions, to claim f is the inverse function of g necessarily implies that g is the inverse function of f, and there is some relationship R between the quantities' values that both f and g are derived from. An implication of this is that anytime we define a representation as capturing f or g, we are implicitly defining a representation that captures g or f, respectively. A created representation for f or g captures R and, necessarily, also g or f, respectively.

Critically, this KDU supports the emergence of mathematical representations (e.g., graphs and equations) as consequences of quantities' covariation (e.g., EGST). It supports students in engaging in EGST because a displayed graph is understood as produced through coordinating quantities' simultaneous variation according to their invariant relationship and the constraints of a coordinate system. Stevens (2019) and Carlson et al. (2022) extended this idea to variables and formulas in order to illustrate that these representations can also emerge as symbolizing quantities' covariation, thus providing evidence that this KDU provides a powerful way of reasoning across representations.

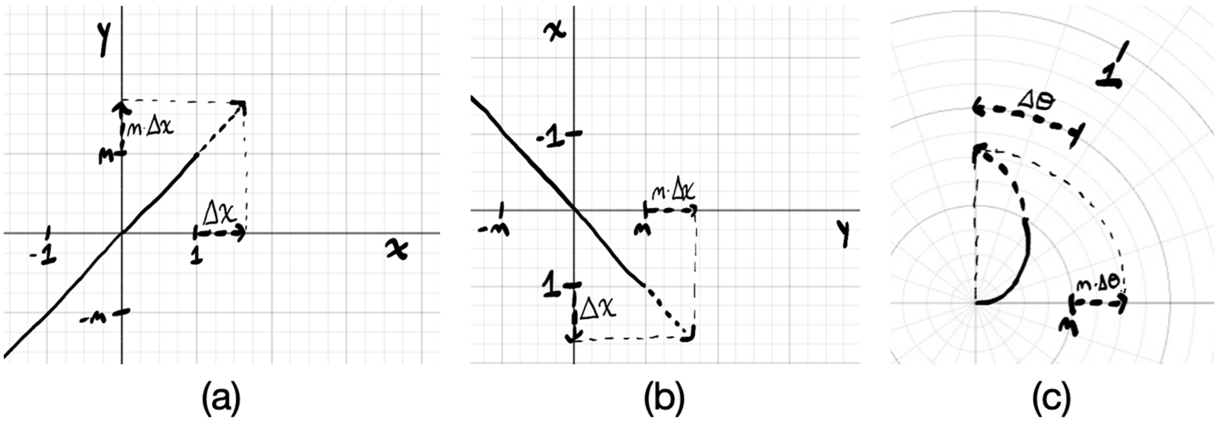

FIGURE 4
Three graphs of a linear, proportional relationship with a rate of change of *m* (or 1/*m*). Each graph emerges through covarying quantities under the constraints of **(a,b)** the Cartesian coordinate system and **(c)** polar coordinate system.

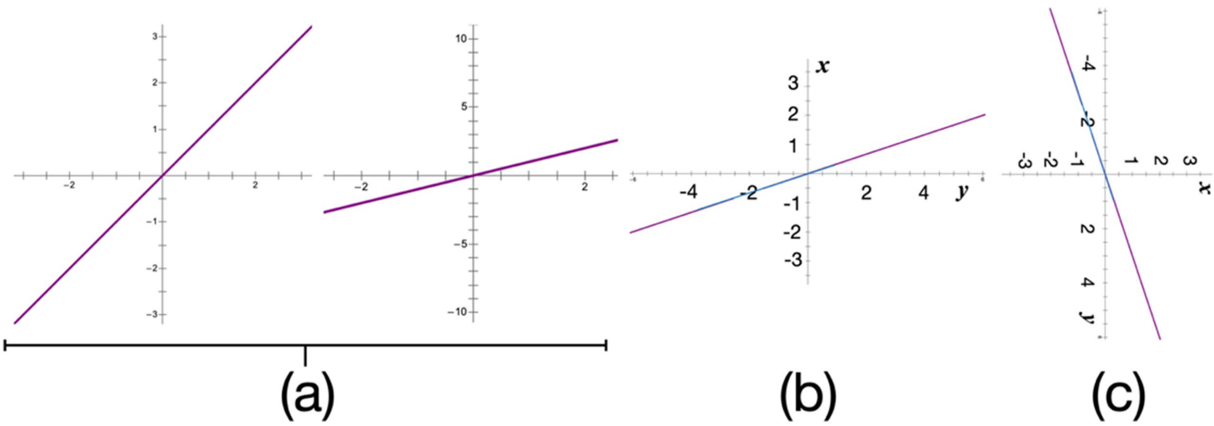

FIGURE 5
(a) Two graphs of $y = x$ and **(b,c)** two graphs of $y = 3x$.

The KDU and linear relationships

The KDU of functions and their graphs as emergent covariational relationships centers a constant rate of change meaning for linear relationships and their graphs. A constant rate of change between two quantities means that as the quantities' magnitudes covary, their amounts of change exist in a proportional relationship. For any arbitrary change x (e.g., Δx), y changes by a scaler factor m of that change (e.g., $m \cdot \Delta x$). If that arbitrary Δx is then scaled by a factor c, the change in y is scaled by the same factor c, yielding a corresponding change in y of $c \cdot m \cdot \Delta x$. With respect to their graphs, this meaning supports understanding a graph as an emergent trace constrained by the proportional covariation. No matter the coordinate system or orientation, a graph is produced by covarying the coordinate system's quantities under this constraint (Figure 4). This is a critical and productive meaning that can act as a KDU as it develops into derivatives and STEM contexts. It is also the reasoning that representations like phase planes and slope fields are built upon. Yet our and others' work with teachers and students suggest that this is not a meaning typically targeted within US schooling (Byerley, 2019; Byerley and Thompson, 2017; Lee et al., 2019; Moore, Silverman et al., 2019; Moore, Stevens et al., 2019; Thompson, 2013; Thompson et al., 2017).

Furthermore, considering the two graphs in Figure 5a, the KDU supports understanding *each* graph as $y = x$ and conveying the same rate of change. Considering Figures 5b,c, the KDU supports understanding that defining each graph as $y = 3x$ necessarily means each graph represents $x = (1/3)y$ (and, for that matter, $y - 3x = 0$ and $3x - y = 0$). It also involves understanding that because each graph conveys that a quantity's change is 3 times as large as another quantity's change, each graph necessarily conveys a quantity's change is 1/3 times as large as another quantity's change. The represented relationship has a rate of change of 3 *and* 1/3. These measures differ numerically because the former is the rate of change of y with respect to x and the latter is the rate of change of x with respect to y. These measures are equivalent quantitatively because they both quantify the rate of change of the underlying relationship; they necessarily and simultaneously imply each other.

The KDU and the sine relationship

Specific to the sine relationship, the KDU of functions and their graphs as emergent covariational relationships also centers an invariant relationship. Stating $y = \sin(x)$ necessarily implies $x = \arcsin(y)$

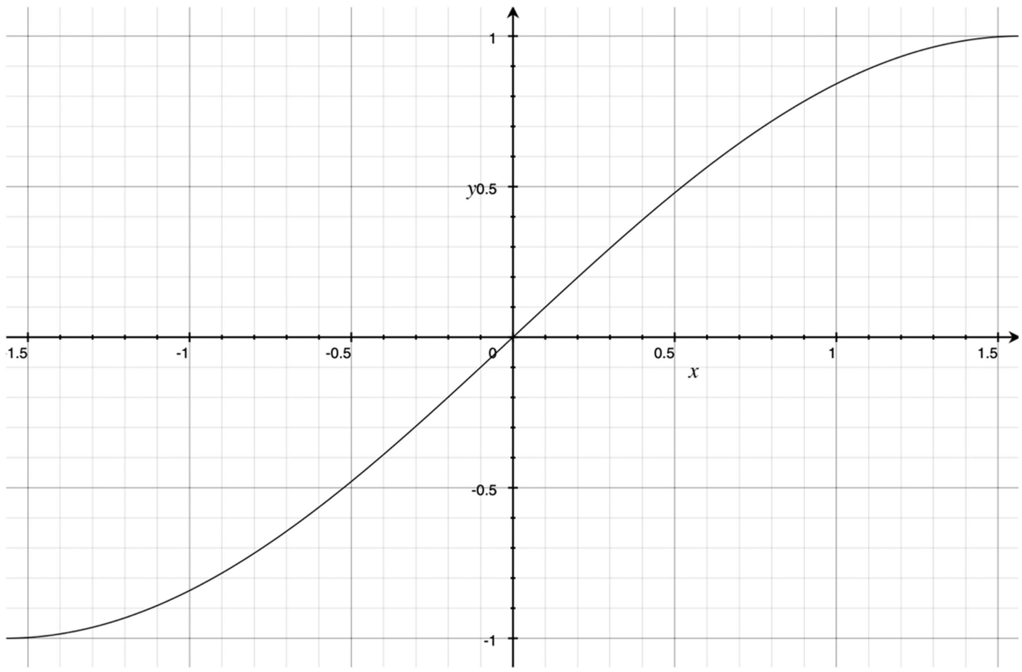

FIGURE 6
A graph of $y = \sin(x)$ and $x = \arcsin(y)$.

and vice versa.[3] Similarly, defining Figure 6 as "The Sine Graph" necessarily implies that Figure 6 is "The Arcsine Graph" and vice versa. In each case, a representation is conceived as emerging in a way that captures the same underlying relationship and, hence, can be defined in terms of both functions. For example, whether considering $y = \sin(x)$ or $x = \arcsin(y)$, one understands that as x increases by successive, constant amounts from 0, y increases by a decreasing magnitude, decreases by an increasing magnitude, decreases by a decreasing magnitude, and increases by an increasing magnitude (Figure 7). This pattern then continues for successive x intervals of 2π. For these same intervals in x, the KDU also involves anticipating that one can consider successive constant increases and decreases in y to illustrate that x increases by an increasing magnitude, increases by a decreasing magnitude, increases by an increasing magnitude, and increases by a decreasing magnitude (Figure 8). This pattern also continues for successive x intervals of 2π. No matter the coordinate system or orientation, constructing a sine and arcsine graph involves covarying the coordinate system's quantities under these constraints.

A disparate extant meaning

The covariation-based meanings outlined above for linear relationships and the sine relationship form alternative meanings (i.e., M_b) that are each an implication of the same KDU. With respect to the competing meanings perspective, it is necessary to clarify an extant meaning with which that KDU can be compared. In our experience, a common extant meaning (i.e., M_a) for function, inverse function, and their graphs involves a collection of shape-based associations, or what Moore and Thompson termed *static (graphical) shape thinking (SGST)* (Moore, 2021; Moore et al., 2019; Moore and Thompson, 2015). They defined, "a student assimilates a displayed graph so that he predicates his actions on perceptual cues and figurative properties of shape…as if [the graph] were a wire" (Moore, 2021, p. 153). With linear relationships, the shape-based meaning consists of learned associations between slope and perceived movement or direction of a line. With respect to broader ideas of function, shape-based meanings consist of a one-to-one association between a function name and a memorized shape. For instance, Figure 6 is strictly *the* sine function because it is the learned shape for the graph of the sine function and thus cannot be a different function (Figure 9); the graph is a named shape. It is necessary that the inverse sine graph be a distinct shape because it is a distinct function. These shape associations are a form of indexical knowledge, which differs from a meaning in which a shape symbolizes attributes of covariational relationships and their graphs' emergence.

In our previous work (Moore, 2021; Moore, Silverman et al., 2019; Moore, Stevens et al., 2019), we have illustrated that shape-based meanings develop in part due to curricula and instruction conflating conventional practices with key aspects of function including quantitative reasoning. Other researchers' findings indicate that this phenomenon is particularly common in the U.S. (Byerley and Thompson, 2017; Thompson et al., 2017). As opposed to positioning representational practices as conventions, common curricular and instructional treatments necessitate maintaining practices such as representing a function's input or x along a horizontal axis. Whether intentional or not, the result of such a treatment is encouraging meanings that do not differentiate between the mathematical properties critical to a topic and the representational properties that are the consequence of a conventional practice. This results in learners constructing a system of meanings such that what should be perceived as representational

3 We remind the reader that we are allowing for multi-valued functions and that today's curricular emphasis on single-valued functions is likely misguided with respect to the cognitive development of the function concept (Thompson and Carlson, 2017; Paoletti and Moore, 2018; Paoletti et al., 2024).

practices are instead perceived as rules and facts to be memorized and unquestionably followed (Moore et al., 2022). In fact, a representational practice can become so entrenched that a student or teacher chooses to deem mathematically-accurate representations incorrect because they do not follow the practice (Moore, Silverman et al., 2019; Moore, Stevens et al., 2019); the shape-based meaning and quantitative-based meaning are disparate.

Enacting the competing meanings perspective

Returning to the question raised in a previous section, in working with individuals holding a shape-based meaning, M_a, as their dominant meaning, we intend to both honor those shape-based meanings while supporting their construction of a quantitative reasoning-based

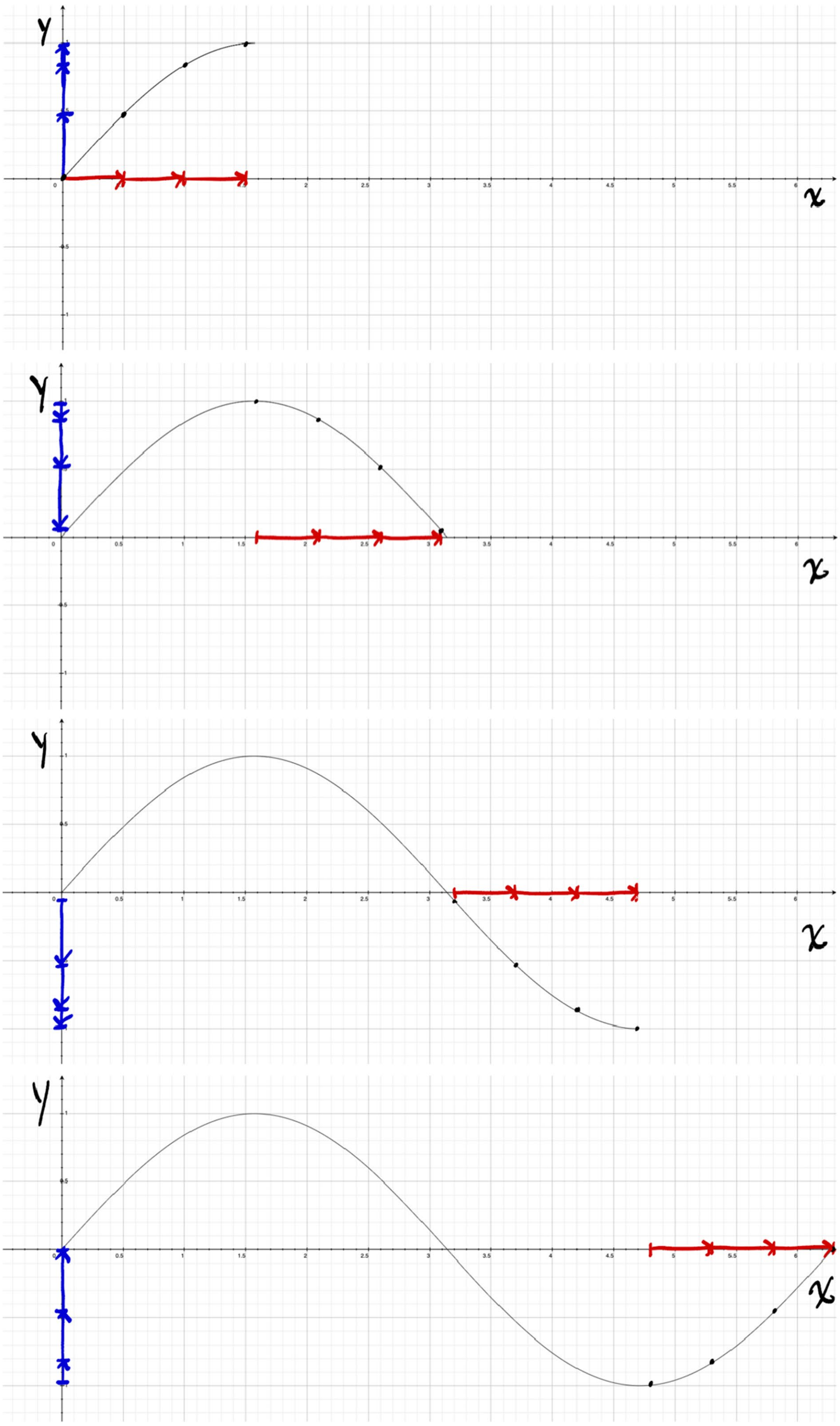

FIGURE 7 (Continued)

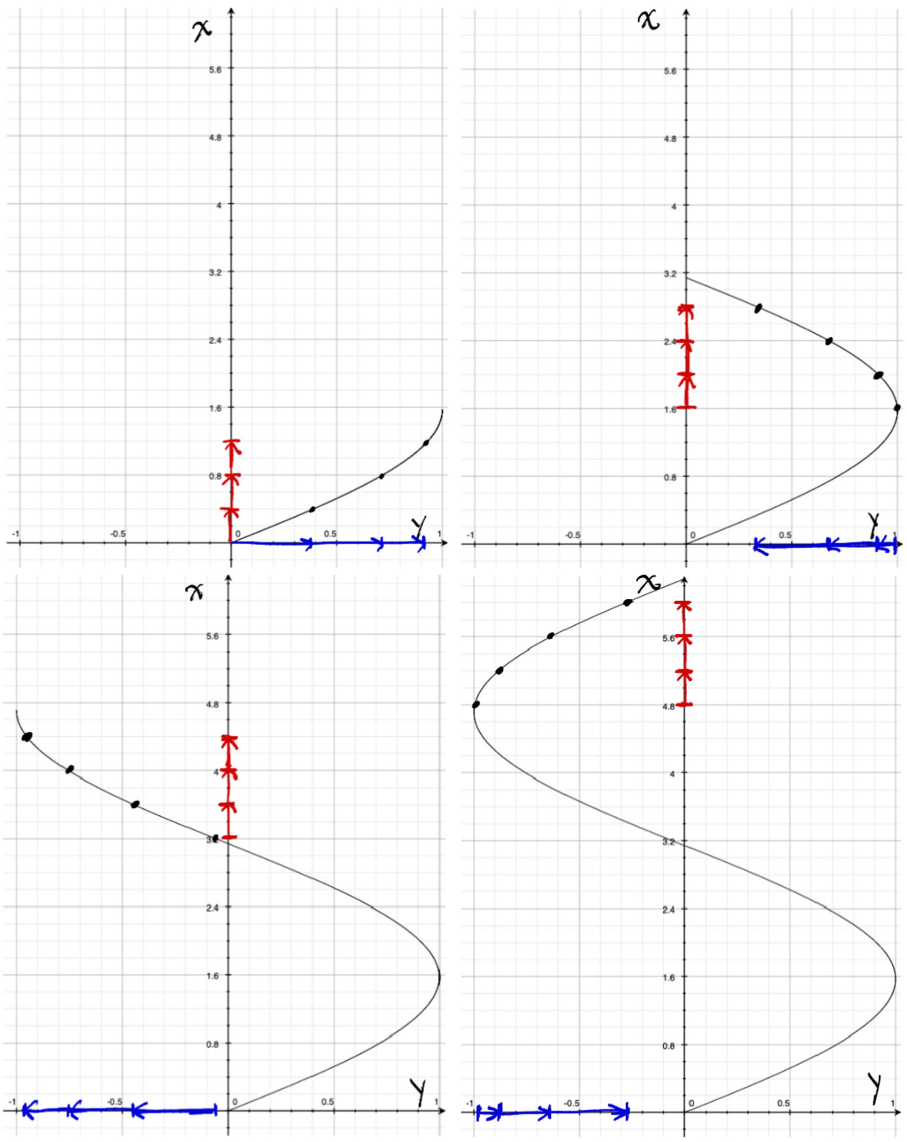

FIGURE 7
Two orientations for illustrating the covariation of the sine relationship for equal changes in *x* with $y = \sin(x)$ and $x = \arcsin(y)$.

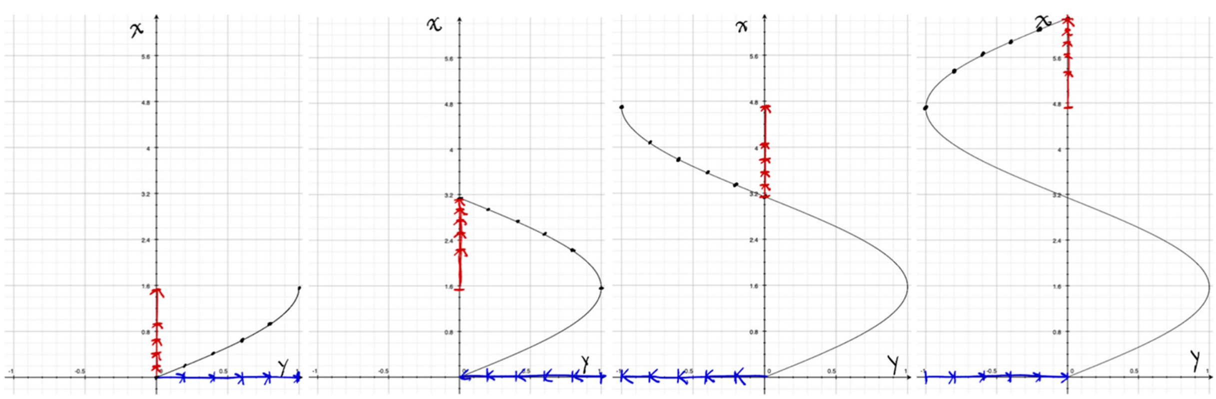

FIGURE 8
One orientation for illustrating the covariation of the sine relationship for equal changes in *y* with $y = \sin(x)$ and $x = \arcsin(y)$.

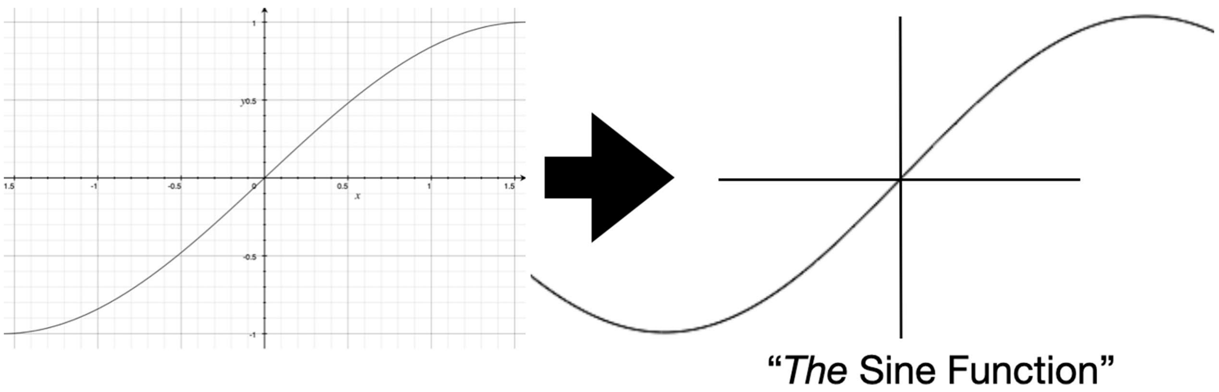

FIGURE 9
Assigning a shape with a learned name.

meaning, M_b. Our goal is also to engage them in learning experiences so that M_b becomes a meaning they view as important and productive for them and their students, and more so than that of the shape-based meaning, M_a. In other words, we hope they construct M_b as a foundational KDU that influences their construction of MKT. In what follows, we discuss enacting the competing meanings perspective in terms of task design principles and goals. We also include brief empirical data to illustrate the perturbations experienced with respect to extant meanings, as well as how the enactment of an alternative meaning can support reconciling a perturbation. The illustrations represent generalizations from a collection of empirical studies with students and teachers. The studies and their methodologies included semi-structured clinical interviews (Ginsburg, 1997) and various forms of teaching experiments (Steffe and Thompson, 2000). They are reported on and summarized in Moore et al. (2022, 2024).

Key principles to the competing meanings perspective

Recall that key cognitive processes for the competing meanings perspective are: (i) the perturbation of an extant meaning, (ii) the reconciliation of that perturbation through the construction of an alternative meaning, and (iii) comparing extant and alternative meanings with respect to the experienced tasks or topics, and potentially with respect to other tasks or topics. In our work, we have found the progression in Figure 10 to be a useful guide in designing learning environments that incorporate these processes.

Before providing an example task sequence, we underscore a few aspects of the progression in Figure 10. With respect to the launch task, it is important that the task is relevant to the teachers' instructional practice. If the launch task is disconnected from their practice or overly contrived, then it is unlikely that they will come to consider any comparison of meanings with respect to the task as relevant to their practice. It is also beneficial if the launch task perturbs an extant meaning in a way that the teacher reaches an impasse trying to reconcile the perturbation. A stronger intellectual need for an alternative meaning emerges if a teacher engages in sustained effort to reconcile the experienced perturbation but remains confident that they should be able to reconcile their perturbation despite not doing so.

With respect to using a sequence of tasks to support an alternative meaning, we acknowledge there are numerous ways an educator can do so. One way to support M_b is to use tasks in which M_b is relevant and M_a is not relevant. Or one might use tasks that target M_b through focused, closed-ended questions and design. To satisfy the competing meanings perspective, however, implemented tasks must generate an intellectual need for M_b that stems from the persistent problematization of M_a. Without persistently problematizing an extant meaning, M_a remains more relevant, familiar, and habitual with respect to teachers' classroom instruction and associated curricular topics.

Our solution to this issue is to use tasks that not only afford both M_a and M_b, but that also draw the meanings M_a and M_b into competition with each other. This is accomplished by implementing tasks in which both meanings remain relevant, enacting M_a yields a perturbation, and enacting M_b does not yield a perturbation. Implementing tasks in which both meanings are relevant but there is an in-the-moment incompatibility between the meanings aids in comparing the meanings. Furthermore, we emphasize that implemented tasks must remain relevant to their classroom instruction. If this is accomplished, the enactment of M_a and its continued perturbation invite further thought as to M_a and M_b's relative relevance not only to the present task, but also to their instruction and student learning.

A design illustration: a competing meanings task sequence

Articulating a task sequence requires that the teacher-educator identify the population with which they are working. A task sequence that achieves the progression in Figure 10 with one teacher population does not imply it will do so with a different teacher population. The following competing meanings task sequence was developed in the context of an undergraduate secondary mathematics teacher preparation program in the Southeast United States (U.S.). The program had several goals related to producing future teacher-leaders including preparing teachers to: (1) center discerning and leveraging student thinking in their instruction; (2) develop and implement transformative content with their students; and (3) act as agents of change within their educational communities. These goals informed the prospective secondary teachers' (PSTs') coursework and field-experiences throughout the multi-year program.

The competing meanings task sequence emerged within a first semester course developed as part of a multi-year, federally funded research project. At its most general level, the project responds to the issue of university mathematics being disconnected from teaching school mathematics. This issue is pervasive,

Perturbing the Extant Meaning

The teacher-educator understands their teachers likely hold M_a as an extant meaning and has a goal of their construction of an alternative meaning M_b.

The teacher-educator uses a launch task that:

1. sensibly affords assimilation to M_a but is designed so that if M_a is enacted, it is highly probable the individual will experience a perturbation;
2. also affords assimilation to M_b and is designed so that if M_b is enacted it is highly probable the individual will not experience a perturbation.

Supporting the Alternative Meaning

The teacher-educator understands M_b, an intended KDU, is likely available to the teacher or within their zone of proximal development.

The teacher-educator uses a task or a sequence of tasks that:

1. engenders and promotes the teachers' repeated enactment of and reflection upon M_b;
2. continues to result in sustained perturbations if M_a is enacted;
3. draws on the teachers' growing awareness of M_b to make explicit key aspects of it.

Prompting the Comparison of Extant and Alternative Meanings

The teacher-educator understands their teachers to have constructed and become aware of M_b so that they can compare it to M_a.

The teacher-educator returns to the launch task to:

1. determine why the enactment of M_a results in a perturbation;
2. determine why the enactment of M_b reconciles that perturbation;
3. analyze the relative affordances and constraints of M_a and M_b.

The teacher-educator returns to other previous tasks or novel tasks and topics to:

1. determine the relevance and implications of M_a and M_b to those tasks and topics;
2. analyze the relative affordances and constraints of M_a and M_b.

FIGURE 10
Illustrating a potential progression for enacting the competing meanings perspective.

and one of its major challenges is identifying how university courses might integrate mathematical and mathematics education content when working with teachers (Wasserman et al., 2023). The project approaches this challenge by investigating how to use research-based models of student thinking to transform PST development so that PSTs' meanings are more sensitive to students' mathematical realities. Reflecting the purpose of the present paper, the major goals of the research-developed course are to perturb and reshape the PSTs' understandings of major algebra and function ideas. The course is also paired with a pedagogy-focused course, which develops the PSTs' capacity to understand, inquire into, and build upon individual student thinking. Together, the two courses challenge the PSTs' extant mathematical meanings and their views on what it means to discern and inquire into student thinking. The PSTs develop a disposition in which they continuously question and improve their own mathematical understandings, while understanding that this pursuit is aided by persistently working to understand and build on their students' mathematical realities.

A launch task

The PSTs are first asked to consider two graphs (Figure 11) and whether they represent $y = 3x$ and "the inverse sine function", respectively. They are asked to be as exhaustive as possible in their assessment. The PSTs predominantly reject both graphs and provide viable justifications. They frequently determine that the two graphs are $y = (1/3)x$ and "the sine graph", respectively, due to aspects of their shape as well as checking various points along the graphs. Following this, a second phase of the task prompts the PSTs to consider a refinement of the two graphs (Figure 12) and whether they represent "$y = 3x$" and "$x = \arcsin(y)$", respectively.

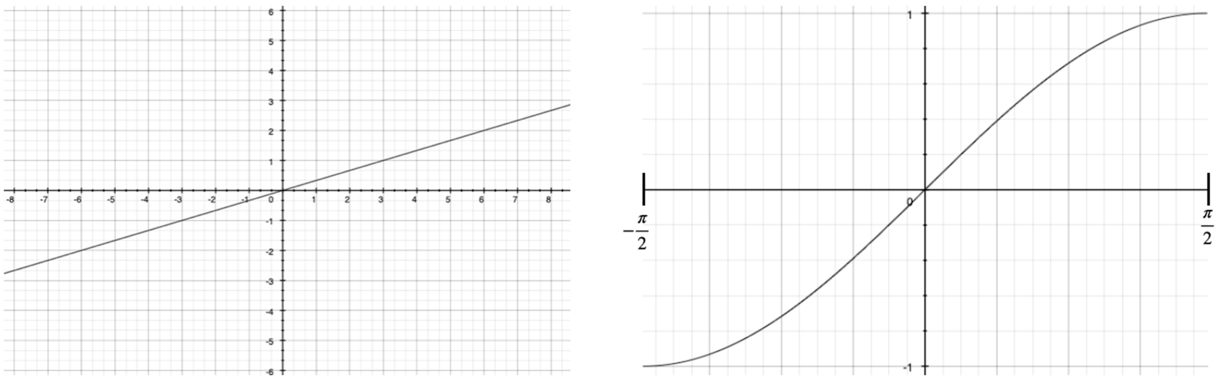

FIGURE 11
Two graphs forming the launch task.

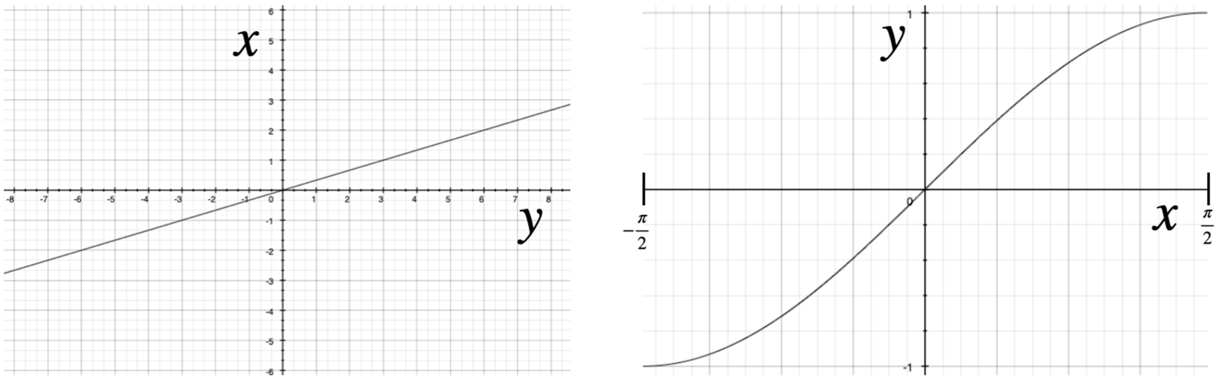

FIGURE 12
Two refined graphs forming the launch task.

This first phase of the task intends to acclimate them to the graphs. The second phase of the launch task is designed to generate a sustained perturbation. The PSTs predominantly determine responses they are satisfied with during the first phase, while the second phase results in them ending the task uncertain whether the graphs represent $y = 3x$ and $x = \arcsin(y)$, respectively. The PSTs reach an impasse for several reasons stemming from their extant meanings. As one reason, in enacting meanings consistent with SGST, they deem Figure 12 (left) to have a negative slope when the graph is rotated so that the x-axis is horizontal; they associate a line sloping downward-to-the-right as necessarily entailing a negative slope. With Figure 12 (right), they deem it to be "the sine graph" because that is the shape they know as "sine." One student explained, "I'm thinking this just kind of looks like the sine graph, like the plain sine graph. Which is going to be different. So, no…".

As another related reason, the PSTs assess that the two graphs must be incorrect because x and a function's input are incorrectly represented, respectively. Yet, with both graphs, the PSTs persistently question their assessment because considering the two graphs point-by-point confirms $y = 3x$ and $x = \arcsin(y)$, respectively, no matter how they rotate the paper upon which the graphs are presented. For instance, one PST identified that Figure 12 (left) rotated 90-degrees is "rising 3 and going [right-to-left] for 1 and so that's still a negative slope of 3." She also identified that the equation $y = 3x$ defines the graph pointwise, adding that the slope implied by that equation contradicts the "negative slope" of the rotated graph. This contradiction resulted in her experiencing an impasse. She ended the task emphatically commenting on this contradiction and impasse, "This is so annoying."

Recall that a launch task should sensibly afford assimilation to M_a but be such that the individual experiences a perturbation if M_a is enacted. This launch task was designed so that if the PSTs assimilated the task to extant meanings involving SGST, and particularly those meanings that *require* maintaining particular graphing conventions, then they will experience a perturbation like those described above. A launch task should also be relevant to the teachers' instructional practice. Here, both graphs involve relationships central to secondary mathematics. Furthermore, the task is not overly complex or contrived in its presentation. The result is that the PSTs view the task as one they should reach a satisfactory conclusion. They thus become confounded by their inability to reach one. We underscore that this impasse is not a negative outcome or some deficiency in the PST. The competing meanings perspective frames the impasse as a catalyst for learning due to creating an intellectual need with the source being the enactment of *the learner's* extant meanings and *their awareness* of an impasse.

A task sequence

Figure 13 outlines the foundation for a task sequence that builds off the launch task described above. The goal of the task sequence is

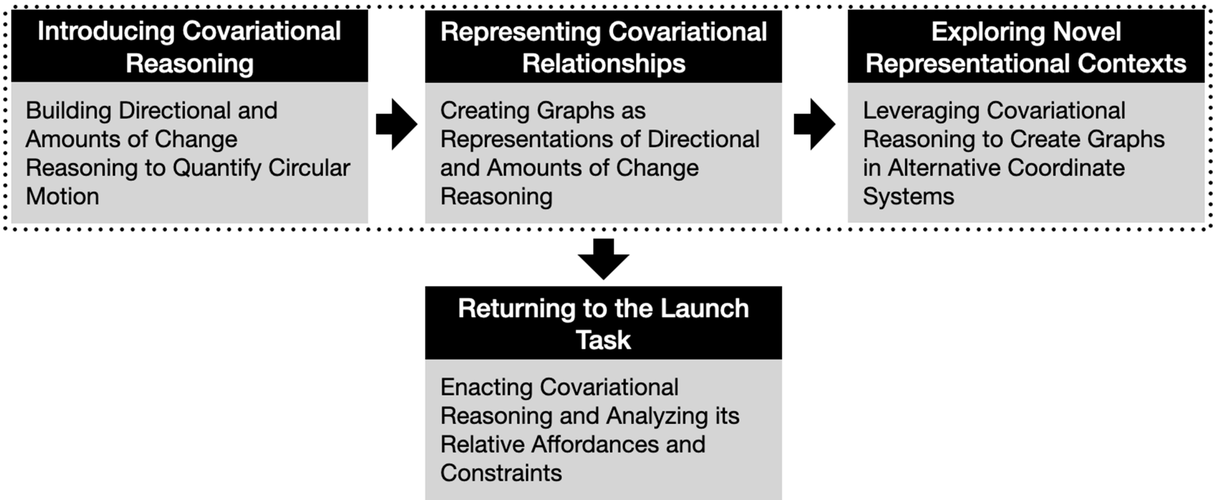

FIGURE 13
A task sequence outline building an alternative meaning.

to build an alternative meaning foregrounding directional and amounts of change covariational reasoning, with graphical representations emerging as products of that reasoning (i.e., EGST). To begin the task sequence, the PSTs explore circular motion contexts such as modeling the trip of a Ferris wheel rider or tracking the position of an edge of a fan or windmill blade. For instance, the PSTs are asked to describe how the distance of a Ferris wheel rider *above the center* of the Ferris wheel varies with respect to the distance the Ferris wheel rider has traveled around the Ferris wheel. The PSTs are also asked to track other variations of these quantities. These variations include alternative distances (e.g., the distance of a Ferris wheel rider *above the ground*) or considering alternative starting points, directions of rotation, or rider speeds.

There are two critical design decisions driving the initial part of the sequence. First, circular motion contexts are common in U. S. secondary mathematics curricula. They are frequently used in trigonometry, geometry, and other modeling contexts, thus maintaining relevance to the PSTs' future practice. Second, we implement this task *without* asking them to create a graph. Rather, the PSTs are only asked to describe and illustrate the covariational relationships using a diagram of circular motion. In our experience, this generates a sustained perturbation due to their attempting to determine relationships using memorized associations between circular motion and graphical shapes, but being unable to describe the covariational relationships underlying those associations. As a result, the PSTs must engage in effortful activity to construct, describe, and illustrate covariational relationships strictly in the context of circular motion (Figure 14).

With one or several covariational relationships identified, the next part of the task sequence has the PSTs creating graphs of those relationships. This provides the PSTs the opportunity to reconstruct the relationships they previously identified but under a different quantitative organization. We typically first have them construct the graphs in a Cartesian coordinate system (CCS). For instance, we ask the PSTs to graph the distance of a Ferris wheel rider *above the center* of the Ferris wheel and the distance the rider has traveled around the Ferris wheel using vertical and horizontal axes, respectively. Through reconstructing the covariational relationship (Figure 15), they can reflect on the mathematical attributes that are present in both the circular motion context and the CCS. In the case of Figures 14, 15, these attributes

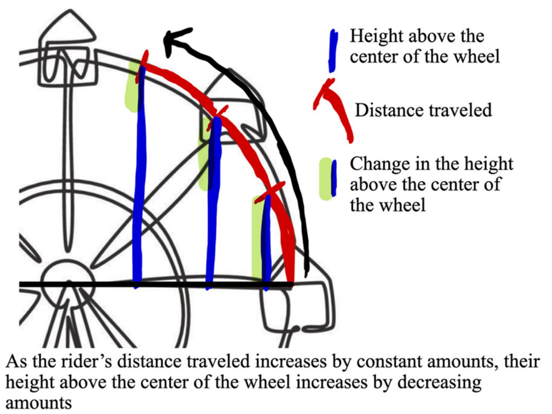

FIGURE 14
Illustrating and describing a covariational relationship for a portion of the ride.

provide the foundation for abstracting the sine relationship. This increases the relevance of the task sequence to their future practice given the centrality of the sine relationship and, more generally, trigonometric relationships in secondary mathematics.

At this point, common instructional approaches might choose to transition to another topic or other relationships. However, our research suggests it is essential to engage in additional explorations that involve alternative and potentially novel coordinate orientations and systems (Moore, 2014; Moore et al., 2022; Moore, Stevens et al., 2019). Following these findings, our task sequence incorporates additional explorations in two ways: (1) in alternative CCS orientations and (2) in alternative coordinate systems.

With respect to alternative CCS orientations, we use the eight possible orthogonal quantitative organizations for two distinct quantities (Figure 16).[4] With respect to alternative coordinate systems, our main explorations have involved the polar coordinate system (PCS). In both

4 It's also an appropriate task to explore non-orthogonal or non-vertical/horizontal organizations.

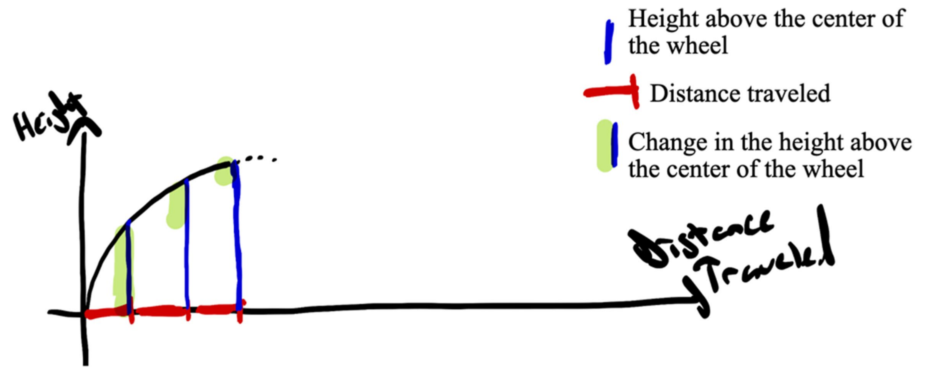

FIGURE 15
Reconstructing the relationship identified in Figure 14 to create a graph.

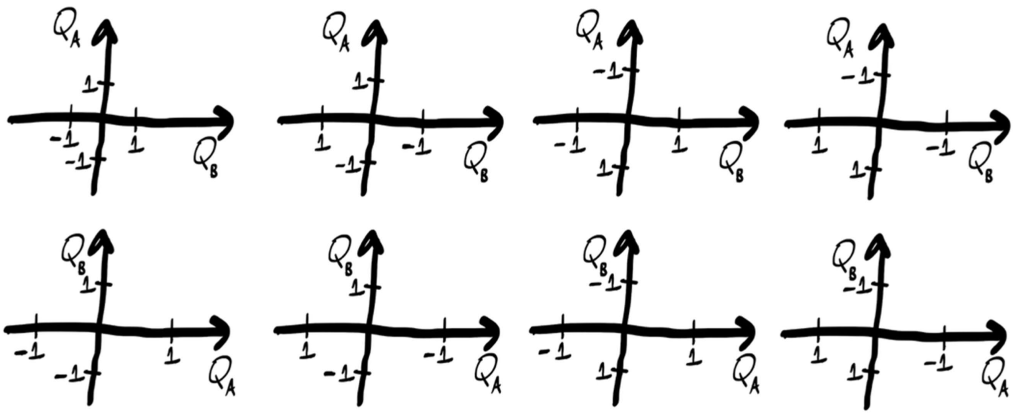

FIGURE 16
The eight CCS quantitative organizations for two distinct quantities.

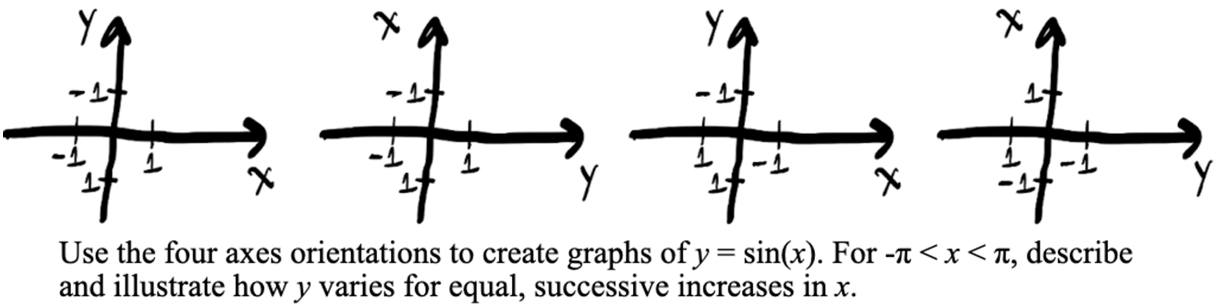

FIGURE 17
A constructing graphs task in alternative CCS orientations.

alternative CCS orientations and the PCS, the PSTs engage in a series of tasks in which they construct graphs to represent covariational relationships or identify the covariational relationships represented by given graphs (see Figures 17, 18 for brief examples). These tasks provide repeated opportunities for the PSTs to enact meanings involving directional and amounts of change reasoning. These tasks also continue to perturb meanings foregrounding SGST. Shape associations continue to be problematized because different orientations and systems produce graphs with different perceptual characteristics. Thus, the tasks afford the PSTs needed opportunities to further differentiate between mathematical properties critical to the relationships they are exploring and those properties that are products of the representational systems they are using. Furthermore, the PSTs continue to perceive these tasks as relevant to their future instructional practice due to their design simply being graphical construction and interpretation tasks in the context of common coordinate systems.

Collectively, the task sequence outlined here centers constructing covariational relationships and reconstructing those relationships in graphical representations. We also invite the PSTs to comment on their progress during the sequence and compare it to

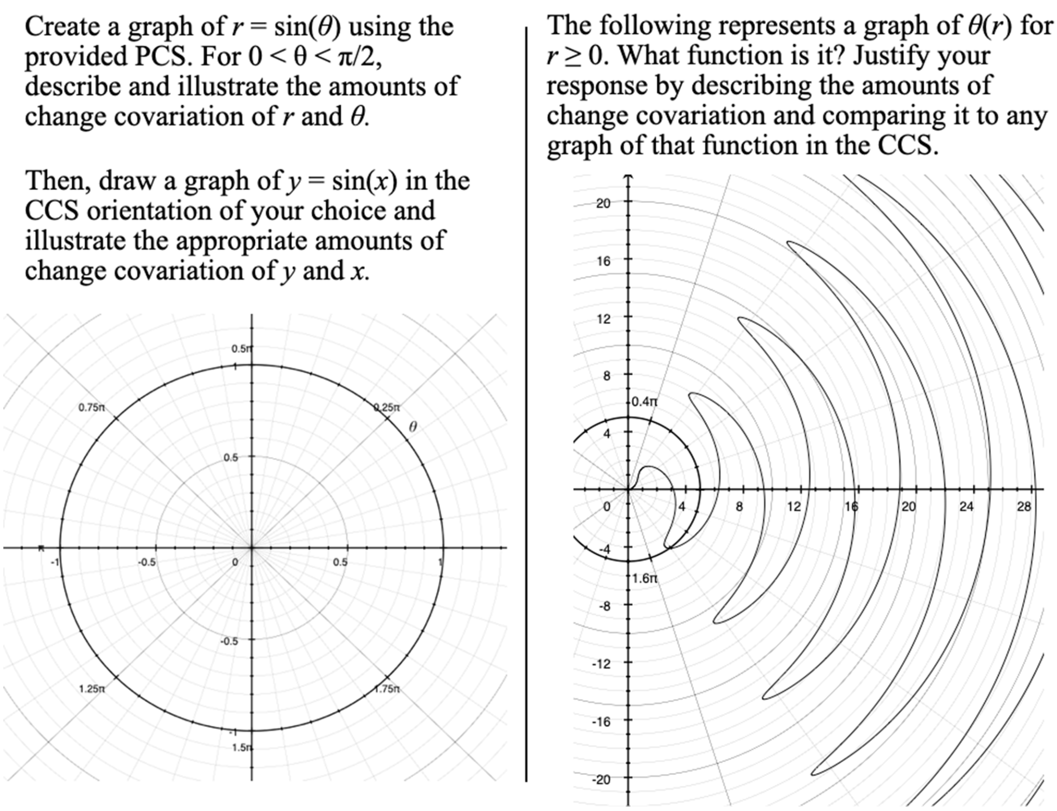

FIGURE 18
A constructing graphs task (left) and identifying a graph task [right, with the solution of $\theta = \sin(r)$] in the PCS.

their previous instruction. At times, we provide reflection assignments with explicit instructions to compare their previous instructional experiences with their current experiences. At other times, they spontaneously raise comparisons and we allow them space to discuss these comparisons as a group. For example, in discussing her perturbations experienced when first trying to construct a graph through recalling a memorized shape, a PST explained, "I can explain it to a point, and then I get like—I confuse myself with the amounts of information I know about a cricle that I was just given to me by a teacher." She then compared that perturbation with her constructing a graph through covariational reasoning. She explained, "So my whole circle talks about width and height and arc, but then this graph itself only talks about arc and height. [*speaking emphatically*] Done it!" Here, the PST drew attention to the importance of understanding that a graph's shape was a consequence of quantities' covariation and could thus be recreated using that covariation. Raising such comparisons between memorizing a shape and producing a graph through covariation provides the PSTs key opportunities to become consciously aware of the affordances and constraints of different meanings. These emerging affordances and constraints can be formalized by returning to the launch task.

Returning to the launch task

Having repeatedly enacted that alternative meaning across a variety of tasks, we return the PSTs to the launch task. We ask them not only to consider the graphs using the ideas explored during the task sequence, but also to reflect on the perturbations from their first engagement with the task. Based on those outcomes, we design subsequent tasks and explorations for the purpose of more directly comparing extant and alternative meanings with respect to each other and other topics.

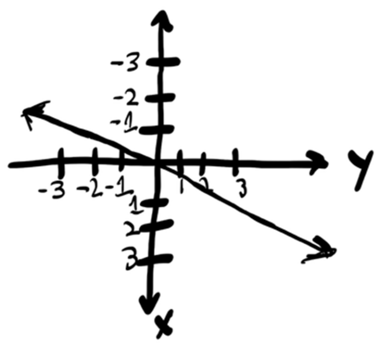

FIGURE 19
A reproduction of the graph created by the PST (Moore, Stevens et al., 2019, p. 13).

We use a PST's response to a task like that described with respect to Figure 11 (left) to illustrate the outcome of PSTs revisiting the launch task. Upon revisiting the task, the PST identified numerous coordinate orientations within which she could graph the relationship $y = 3x$. In response, we asked her about her graph that was sloping downward left-to-right and someone claiming it to have a negative slope (Figure 19). As reported in Moore, Stevens et al. (2019), she explained,

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> You'd have to notice that even though it looks like a negative slope [*making a hand motion down and to the right*] because we call it slope because it's visual and it's easy to visualize a negative and positive slope [*making hand motions to indicate different slopes*]. But that's only visual on our conventions of how we set it up. Um, but like [*pointing to the graph*] if slope is rate of change we can still see that for like equal increases of x [*making hand motions to indicate equal magnitude increases*] we have an equal increase of y [*making hand motions to indicate equal magnitude increases*] of three. And so for equal positive increase of one [*sweeping fingers vertically downward to indicate an increase of one*], we have an equal positive increase of three [*sweeping fingers horizontally rightward to indicate an increase of three*]. And so it is a positive slope (Moore, Stevens et al., 2019, p. 12).

In the quote above, the PST directly references both the extant and alternative meanings. Her positioning of the two meanings suggests she has concluded that if one defines slope in terms of rate of change, then slope is subordinate to the quantitative properties of rate of change. That is, rate of change is a multiplicative relationship between covarying quantities' changes and graphically that is captured by the emergence of the graph to represent that relationship under the constraints of the coordinate system. Because slope is subordinate to this, she contends that classifying slope is dependent on and thus subordinate to properties of the quantities' covariation; if conventions are followed, this allows for learned visual inferences, but the legitimacy of those inferences remains a consequence of rate of change attributes.

As another example, we turn to a PST's response to the task described with respect to Figure 11 (right). As reported in Moore (2021), after having identified that the graph can represent the sine or arcsine functions depending on input and output choices for the axes, we presented her the graph in Figure 20 along with a claim of it being a graph of the inverse sine function instead of the given graph. She identified that both graphs are the same relationship and explained,

> …you could show the increasing, right. So I mean you could just like disregard the y and x for a minute, and just look at, like, angle measures. So it's like here [*referring to* Figure 20], with equal changes of angle measures [*denoting equal changes along the vertical axis*] my vertical distance is increasing at a decreasing rate [*tracing curve*]. And then show them here [*referring to* Figure 11 (right)] it's doing the exact same thing. With equal changes of angle measures [*denoting equal changes along the horizontal axis*] my vertical distance is increasing at a decreasing rate [*tracing curve*]…So even though the curves, like, this one looks like it's concave up [*referring to* Figure 20 *from* $0 < x < 1$] and this one concave down [*referring to* Figure 11 *from* $0 < x < \pi/2$], it's still showing the same thing (Moore, 2021, p. 59).

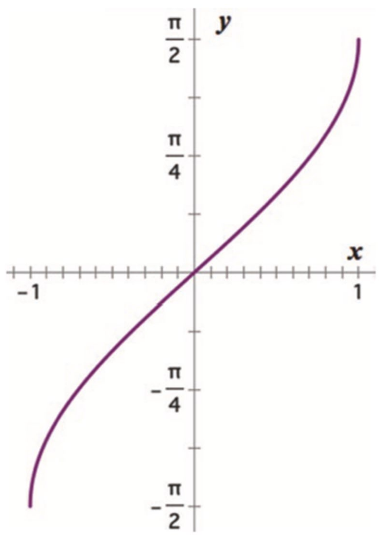

FIGURE 20
A graph of $x = \sin(y)$ and $y = \arcsin(x)$.

Consistent with the previous PST's actions, the PST directly references extant and alternative meanings. Here, the PST positions the two meanings so that the associations between concavity, rate of change, and function names are subordinate to covarying quantities (see her illustration in Figure 21). Whereas traditional curricular approaches promote the meaning of "concave up" as strictly "increasing rate," the prospective teacher problematized this meaning by identifying that both concave up and concave down curves convey a quantity increasing at a decreasing rate with respect to another quantity. Alternatively, such reasoning enables understanding that if a concave down curve conveys y increasing at a decreasing rate with respect to x, then the curve conveys x increasing at an increasing rate with respect to y.

The PSTs' actions illustrate how returning to the launch task provides an opportunity for a PST to situate extant and alternative meanings with respect to each other. In these cases, the PSTs foregrounded quantitative reasoning-based meanings, while understanding any shape-based associations are subordinate to those and constrained to representational conventions. The PSTs' abilities to differentiate between the mathematical properties critical to a concept from those that are merely products of a representational practice is a hallmark of the KDU targeted in this paper. With that said, we do not present these two cases to argue that either PST had constructed a KDU. Any such claims would require additional evidence as to the extent the PSTs' meaning enables them to connect various important mathematical ideas and topics. But the foundations for a KDU that reflects research-based models of student thinking are in place.

Some considerations in enacting the competing meanings perspective

The competing meanings perspective hinges on perturbing extant meanings in ways that lead to the construction and privileging of alternative meanings. This is a difficult task for numerous reasons, including the complexity of working with teachers who operate in intersecting societal, institutional, and mathematical contexts. Whether pre- or in-service, teachers have had years of experience developing beliefs, knowledge, and images of their students, content, instruction, and institutions (Buchbinder et al., 2019; Erickson et al., 2021). Furthermore, teachers often experience professional development that is perceived as disjointed from their practice and the students in their classroom, particularly at the university level (Wasserman et al., 2023). Thus, many challenges can arise when trying to impact teachers' practice, regardless of a teacher-educator's focus.

We identified above two ways in which we attempt to mitigate these challenges. First, we accept at the onset that the development of an alternative meaning that becomes a KDU is a process that occurs

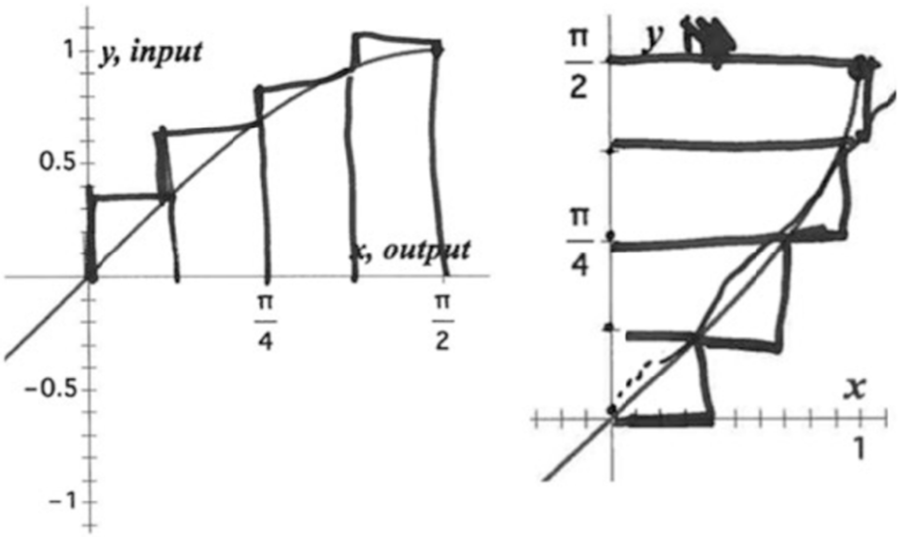

FIGURE 21
The prospective teacher illustrating the invariant covariational relationship (Moore, 2021, p. 59).

over time. A sequence of tasks occurring across multiple days, weeks, or even months are often necessary for the progression in Figure 10. Second, we underscore the two-sided importance of teachers experiencing perturbed extant meanings but within the context of tasks that remain relevant to their teaching. If a task perturbs their extant meanings but is deemed irrelevant to their teaching, the teacher is unlikely to give serious attention to their perturbation. Yet if the task is deemed relevant to their teaching, then their perturbation hopefully sparks curiousity and a desire to reconcile that perturbation.

As third way in which we attempt to mitigate the challenge of perturbing teachers' meanings while encouraging their construction of an alternative meaning is to use contexts or topics within which teachers are likely to perceive a need to improve their knowledge and instruction. During pilot work of the project informing this paper, we initially implemented a task sequence using linear relationships. This was problematic in two ways. First, the teachers had a sophisticated array of meanings at their availability to solve problems involving linear relationships. Thus, the problems necessary to perturb them were often overly contrived or disjointed from their practice. Any developed alternative meaning was thus viewed as overly complex or unnecessary for secondary mathematics. Second, albeit related to the previous point, the experienced perturbations left the teachers frustrated. This feeling was exacerbated when reconciling that perturbation involved an alternative meaning they perceived as disjointed from their practice.

Our response to this challenge was to shift our initial focus to circular motion, angle measure, and trigonometric relationships. With the PST population referenced above as well as secondary teachers, we have found teachers to be more readily perturbed when engaged with these topics than with linear relationships. Critically, we have found teachers to be more comfortable and transparent with their perturbations. Together, these two aspects tend to create a greater and more embraced intellectual need for alternative meanings. As teacher-educators, we can then target the development of those alternative meanings before turning the teachers' attention to the implications of those meanings for topics like linear relationships. More generally, a teacher-educator looking to incorporate a competing meanings perspective should identify those topics in which teachers perceive a need to improve their knowledge and instruction. Then, after the intended alternative meaning is explored, the teacher-educator can turn to related topics in order to explore the implications of that meaning.

Focusing on an area teachers are likely to perceive a need for improvement can, however, present a challenge. Their experienced need for improvement can stem from their meanings for other related topics. Thus, supporting a teacher in developing an alternative meaning might require a teacher-educator to first support the development of meanings for other related ideas. If they do not, then the teachers can experience much difficulty and frustration trying to construct an alternative meaning, and especially one that is relevant to their instruction. In our work, our focus on circular motion has required that we focus on developing specific meanings for angle measure that lend themselves to a quantitative reasoning-based approach to trigonometric relationships (Moore, 2012, 2013; Thompson et al., 2007). Relatedly, our focus on the PCS requires an instructional sequence that has the PSTs construct the quantitative organization of the PCS before considering PCS graphs (see Moore et al., 2013, 2014). Although this requires instructional time, an additional implication is that the focus on angle measure provides a connecting thread across circular motion, trigonometric relationships, and the PCS. The PSTs are able to explore how a particular alternative meaning for angle measure acts as a KDU in that it affords powerful meanings for subsequent topics that would not be available in its absence.

To close this section, we admit that designing a task that generates the initial perturbation targeted by the competing meanings perspective is difficult, especially when the alternative meaning is disparate from the extant meaning. One way we have successfully navigated this challenge is to design launch tasks that incorporate student work—whether hypothetical or drawn from our empirical research—that was a product of the alternative meaning and perturbs the extant meaning. For instance, we have presented Figures 11, 12 as graphical solutions produced by students and prompted the PSTs to comment on the viability of those solutions and the thinking that might have produced them (Moore, 2021; Moore, Silverman et al., 2019). Such prompts frequently result in teachers experiencing an initial perturbation and accompanying intellectual need sufficient for the next process of the competing meanings perspective. Namely, they are left puzzled as to the student thinking that might have generated the solutions and whether or not the solutions and reasoning are viable.

There is an important qualification in our use of student work to perturb an extant meaning during a launch task. We do not imply such use, even when authentic student work, results in the teachers perceiving the solution or associated thinking as immediately relevant to their instruction. In fact, we consider such expectations unwarranted. Buchbinder et al. (2019) illustrated as much by identifying that teachers draw on multiple resources to understand and appraise student solutions and thinking. Thus, the goal of using hypothetical or authentic student work during a launch task is merely to occasion a perturbation that can afford the construction of an alternative meaning that researchers have identified as a KDU. After the teachers develop that alternative meaning during subsequent tasks, they can revisit the student solutions and potential thinking driving those solutions. In doing so, their construction of the alternative meanings enables them to not only understand the posed student reasoning as viable, but also consider the relevance of such a meaning for their instruction and supporting students' mathematical development. Such relevance is key to connecting the student work and underlying mathematics to their practice, and should thus be a persistent principle of teacher development (Wasserman et al., 2022). Returning the teachers to hypothetical work provides them an opportunity to envision their alternative meanings in terms of the work and reasoning of their students.

Closing

In this paper, we presented a learning process that identifies how two meanings might be brought into comparison via processes of assimilation, accommodation, and perturbation. This was for the purpose of supporting teachers' meaning development from extant meanings to those meanings that better reflect research-based models of students' mathematical realities. Furthermore, we illustrated how such processes involve different forms of intellectual need, including that with respect to the activity of a task, the comparison of meanings with respect to a task, and the comparison of meanings with respect to the broader implications of those meanings. We also provided a task sequence to illustrate the cognitive processes defined by the competing meanings perspective, as well as various task-design principles associated with the perspective. Despite being an outgrowth from a body of empirical and theoretical work, the competing meanings perspective is still in its infancy as a theory. Moving forward, we envision a need for further connecting to other extant constructs and perspectives, the results of which will continue to shape and develop the competing meanings perspective. Additionally, future research must examine the instructional and task environments that afford the identified cognitive processes in diverse teacher populations and authentic instructional contexts.

A learner likely needs to experience numerous, repeated opportunities to reconstruct and reflect upon meanings in order to construct a KDU (Simon, 2006). We present the competing meanings perspective as a three-phase process above, but the reader should conceive the processes of the competing meanings perspective as needing to occur numerous times for a meaning to be transformed into a KDU. Furthermore, cognitive development is complex, and we do not mean to imply that the processes above necessarily occur linearly. This is especially true with respect to exploring the implications of multiple meanings. Such explorations can, at any moment, prompt returning to comparisons of each meaning in the context of any previously experienced task. Much work is left to be done to understand how the processes associated with competing meanings unfold to the benefit of teachers and students. Furthermore, understanding relationships between the cognitive processes outlined here and the different resources teachers draw on to understand student thinking as identified by Buchbinder et al. (2019) offers a fruitful ground for future research. In fact, we envision that instructional sequences informed by the competing meanings perspective can form some of the alternative instructional exchanges necessary for reform goals.

We focused exclusively on the competing meanings perspective in relation to a particular KDU. This was both a pragmatic and necessary decision. Pragmatically, this enabled discussing competing meanings with specific examples. The competing meanings perspective foregrounds mathematical meanings and their interaction, and we would have failed our epistemological underpinnings had we avoided discussing particular meanings in detail (Thompson, 2013). At the same time, focusing on the KDU we did here was unavoidable. The KDU emerged out of our research program and, hence, we produced the competing meanings perspective as a generalization of our experiences working with teachers (and students) to support that KDU. Everything described here emerged from experiences with students and teachers and are thus best envisioned as co-constructed by them. For instance, all tasks discussed in this paper are a direct reflection of empirical models of students' and teachers' mathematics. Furthermore, it was through the generation and use of those tasks that we were able to empirically investigate the competing meanings perspective. We thus envision that any articulation of the competing meanings perspective in relation to another KDU will require empirical work to construct the meanings constituting that KDU, the extant meanings individuals hold, and how to create desired interactions between those extant and alternative meanings.

To date, we have concentrated much of our research focus on the first two aspects competing meanings and relatively less on the ways in which teachers compare extant and alternative meanings (cf. Paoletti, 2020). A meaningful, reflective comparison of meanings is a developmental process that occurs across a sequence of experiences that engender the processes detailed by competing meanings. Thus, we envision a fruitful area of inquiry to be more detailed and nuanced investigations of such a developmental process. Similarly, our primary focus in this paper was on the development of personal meanings that reflect research-based KDUs and models of student thinking. An additional fruitful area of inquiry will be further explorations of how those personal meanings and KDUs are further transformed into MKT so that they have pedagogical power (see Tallman et al., 2024).

We agree with researchers who have argued that processes of reflecting/reflected abstraction are necessary for teachers to transform their own personal knowledge including KDUs into knowledge that has pedagogical power (i.e., MKT) (Liang, 2021; Silverman and Thompson, 2008; Tallman and O'Bryan, 2024; Tallman et al., 2024). With respect to constructing MKT, Tallman et al. (2024) explained, "a teacher cognizant of the mental actions and operations that comprise their mathematical schemes is positioned to reflect on the conceptual *process* by which students might construct similar meanings, and to enact pedagogies to engender this constructive process" (p. 23). In this paper, we do not make strong claims regarding the development of MKT, as there are numerous factors other than a teacher's MKT that mitigates their instructional practices and the meanings they target in their classroom. The cognitive processes associated with the competing meanings perspective merely form a foundational basis for productive MKT. But, based on literature identifying the critical role of meanings and teachers engaging in processes of reflecting/reflected abstraction, we view the competing meanings perspective as one potential tool to investigate abstraction processes and the ways in which they shape teachers' pedagogies, interactions with students, and, ultimately, MKT.

Data availability statement

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

Ethics statement

The studies involving humans were approved by the UGA Institutional Review Board. The studies were conducted in accordance with the local legislation and institutional requirements. The participants provided their written informed consent to participate in this study.

Author contributions

KM: Project administration, Methodology, Supervision, Formal analysis, Data curation, Visualization, Conceptualization, Funding acquisition, Writing – review & editing, Investigation, Resources, Writing – original draft. HT: Writing – review & editing, Methodology, Investigation, Writing – original draft, Conceptualization, Formal analysis. IS: Writing – review & editing, Conceptualization, Investigation, Writing – original draft, Formal analysis, Methodology. BL: Writing – review & editing, Methodology, Investigation, Writing – original draft, Conceptualization, Formal analysis.

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Appendix: glossary of terms.

Term	Description	Support source
Assimilation	Imbuing one or several meanings to an experience	Piaget (2001)
Perturbation	A cognitive state that occurs when assimilation results in an unexpected result	Piaget (2001)
Accommodation	Reconciling a perturbation through assimilation to novel or alternative meanings	Piaget (2001)
Scheme	A conceptual structure constituted by a perceived situation, goal, generated activity or action, and result	Steffe (2024)
Meaning	An implicative system of schemes that is constructed to assimilate an experience	Thompson (2013)
Quantitative reasoning	Constructing and reasoning about measurable attributes and relationships between them	Thompson (1993)
Covariational reasoning	Quantitative reasoning that involves constructing and reasoning about quantities varying in tandem	Thompson and Carlson (2017)
Directional covariation	Reasoning about increases/decreases/constancy in one quantity as another quantity increases/decreases/remains constant	Carlson et al. (2002)
Amounts of change covariation	Reasoning about variation of increases/decreases/constancy in one quantity as another quantity increases/decreases/remains constant	Carlson et al. (2002)
Intellectual need	A cognitive state of perturbation that necessitates an educator's construction of alternative meanings that will reconcile that perturbation	Harel (2013)
Epistemological justification	The knowledge constructed when an individual becomes explicitly aware of how a constructed meaning reconciled a perturbation	Harel (2008)
Mathematical knowledge for teaching (MKT)	A developmental, dynamic system meanings that involves the construction of personal meanings to the transformation of those meanings so they have pedagogical power and sensitivity to students' mathematical realities	Silverman and Thompson (2008)
Key developmental understanding (KDU)	A meaning identified as pivotal in students' mathematical development and thus important for a teacher to hold	Simon (2006)