The Circle Approach to Trigonometry

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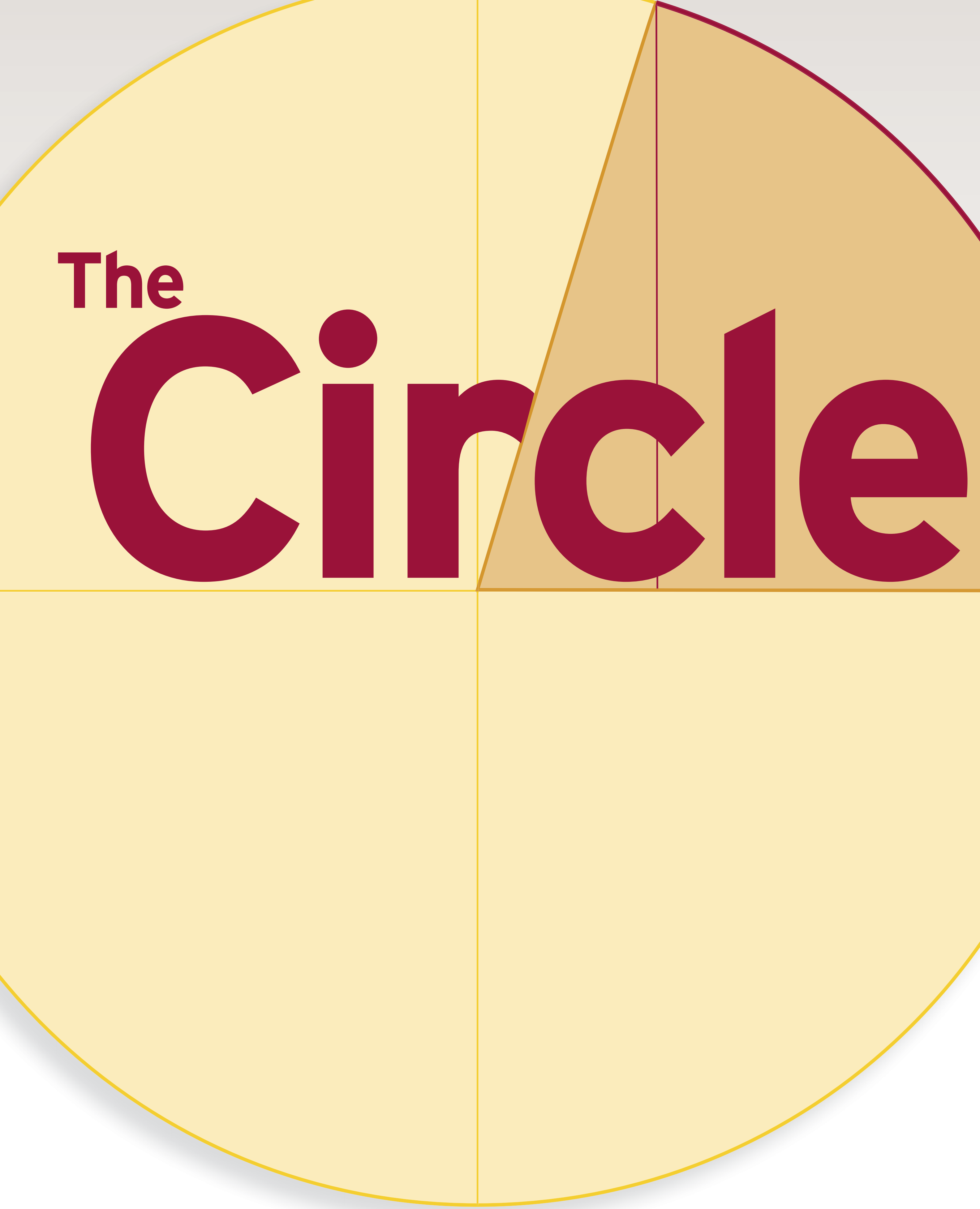
The Circle



A connected introduction of angle measure and the sine function entails quantitative reasoning.

Approach to Trigonometry

Kevin C. Moore and Kevin R. LaForest

How do your students think about an angle measure of ninety degrees? How do they think about ratios and values on the unit circle? How might angle measure be used to connect right-triangle trigonometry and circular functions? And why might asking these questions be important when introducing trigonometric functions to your students?

When teaching trigonometric functions, most teachers would agree that students have difficulty using trigonometric functions to relate quantities in circle and right-triangle contexts. Research shows that the ways in which students understand angle measure do not support thinking about trigonometric functions as relations between two quantities (Akkoc 2008; Hertel and Cullen 2011; Moore 2012; Thompson, Carlson, and Silverman 2007; Weber 2005, 2008). As a result, students struggle to use these functions in novel contexts in their future mathematics, science, and engineering classes.

If students are to use trigonometric functions productively, they must understand angle measure, the unit circle, and right triangles in ways that let them see trigonometric functions as relations between two quantities. In addition, trigonometry instruction should adhere to the Common Core State Standards for Mathematics' emphasis on students' quantitative reasoning (CCSSI 2010). In this article, we draw from recent research on student learning in trigonometry to illustrate a connected introduction of angle measure and the sine function that entails quantitative reasoning.

To demonstrate these ideas in practice, we provide examples of precalculus students' solutions to various trigonometry problems. The ideas presented here also outline nuances involved in increasing student learning during a sequence of activities like that presented by Landers (2013). In particular, we draw attention to reasoning about varying quantities in the context of graphing and how issues of measurement help students understand the unit circle and ratios.

ANGLES AND MEASURING ARCS

Students often think about radian measure differently from the way that they think about degree measure. This results in a dichotomy in their understanding of trigonometric functions. As Bressoud (2010) and Thompson (2008) noted, traditional approaches to trigonometry and angle measure contribute to students' difficulties with the subject. Students' first experiences with angles often involve using a prelabeled protractor to determine their measure, classify them as acute or obtuse, and relate them as supplementary or complementary. This approach facilitates skill with a protractor and calculations relating angle measures but fails to help students develop an understanding of the fundamental basis for the act of measuring an angle; little attention is given to the fact that a protractor's design is based on the partition of an arc or to the question of how we might measure an angle without a protractor.

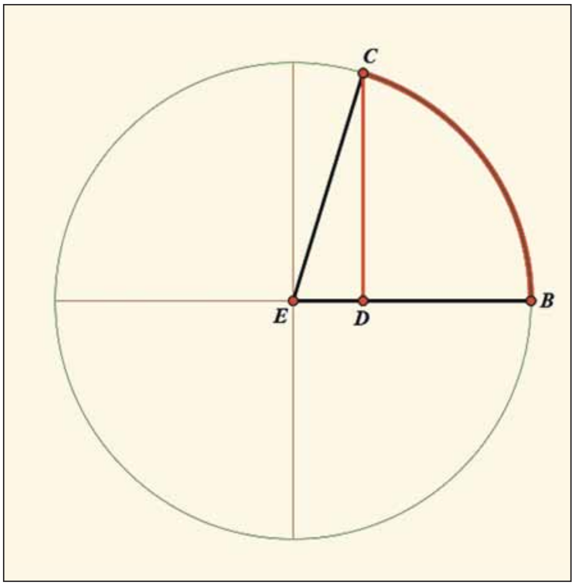


Fig. 1 Students should visualize a circle and an increasing arc *BC*.

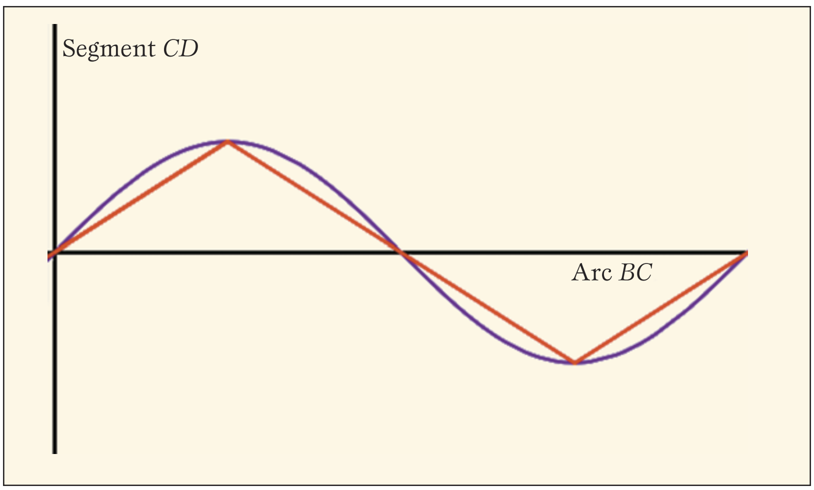


Fig. 2 These two graphs represent the same directional covariation.

Whereas common approaches to degree angle measures do not fundamentally involve measuring arcs, radian measure is explicitly tied to arcs. Any circle's circumference is 2π radius lengths (radii). For any angle and a circle centered at its vertex, an angle measured in radians conveys an arc length that measures a particular number of radii. The size of the circle does not matter. If we connect angles measured in radians to arc lengths measured in radii, the standard formula $\theta = s/r$ emerges. The ratio s/r represents the multiplicative relationship between the arc length and the radius.

Because degrees and radians measure the same quantity—that is, the "openness" of an angle—and are both used in trigonometry, students must understand both units in compatible ways. One approach is to treat angle measures, regardless of unit, as conveying the fractional amount of a circle's circumference that subtends the angle. That relationship holds for all circles centered at the vertex of the angle. An angle measure of 1 radian implies that the angle is subtended by an arc $1/(2\pi)$ of a circle's circumference; an angle measure of 1 degree implies that the angle is subtended by an arc 1/360 of a circle's circumference. Just as a foot is a scaled version of a meter, a degree is a scaled version of a radian.

In the following excerpt of student dialogue, a student discusses connecting angle measure to subtending arcs on circles with radii of 2 inches, 2.4 inches, and 2.9 inches. The student reasons about measuring in radii to determine the arc lengths, and his angle conversion method illustrates understanding angle measure, regardless of unit, as a fractional amount of a circle's circumference.

Student: A very easy way [of converting degrees to radians] is putting 35 over 360 is equal to *x* over 2 pi . . . With that, all I have to do is just multiply the answer by 2 inches, 2.4 inches, and 2.9 inches to get the different arc lengths [*tracing each arc length*] right there, because radians is just a percentage of a radius . . . What you're doing [*describing angle conversion*] is just technically finding a percentage. Like 35/360 is 9.7 percent of the full circumference.

COVARIATIONAL REASONING AND THE SINE FUNCTION

Developing the same process to generate degree and radian measures provides one argument for taking an arc approach to angle measure. A second rationale is that an arc approach to angle measure supports a covariational understanding of trigonometric functions. A growing body of literature identifies covariational reasoning—coordinating how two quantities vary in tandem—as critical for students' success in secondary school mathematics and in learning trigonometry, particularly when modeling situations like those presented by Landers (2013). Carlson and colleagues' (2002) extensive discussion of covariational reasoning showed that, unfortunately, covariational reasoning does not receive significant attention in U.S. curricula.

To demonstrate a covariational approach to developing the sine function, consider a circle with a point *C* traveling counterclockwise from the 3 o'clock position (see **fig. 1**). As *C* moves counterclockwise along the circle, the length of arc *BC* increases. We can also reason that as arc *BC* increases in the first quadrant, so does the vertical distance of point *C* above the horizontal diameter. This directed length (positive lengths above and negative lengths below the horizontal diameter) is illustrated by segment *CD*. Over the next two quarters of a rotation, the length of arc *BC* continues to increase while segment *CD* decreases. Over the last quarter of a rotation, arc *BC* increases in length and segment *CD* increases (i.e., becomes less negative).

Watching *CD* increase or decrease as arc *BC* increases, we consider the directional covariation of the two quantities. Multiple graphs can convey the same directional relationship (see **fig. 2**), but only one of these graphs, a sine function, conveys the correct relationship between arc length *BC* and segment length *CD* when rate of change and amounts of change are taken into account. To gain a better understanding of how the two quantities covary, we compare changes over the first quarter of a rotation. Doing so, we identify decreasing increments in segment length *CD* for successive equal changes in the arc *BC* (see **fig. 3**). Similar reasoning can be used to represent the relationship over an entire rotation.

A graph that reflects the described covariational relationship (see **fig. 4**) resembles the graph of the sine function. Although the graph in **figure 4** is not technically the sine function (the following sections address this point), students must understand the graph as a covariational relationship if they are to understand the sine function as a quantitative relationship. Instead of a ratio in a static right triangle with a restricted domain of angle measures, a circle context better enables the student to think about the sine function in terms of two quantities that covary in a periodic manner. Also, when angle measures are associated with arcs, students can connect a varying arc to an angle of rotation.

To further illustrate covariational reasoning, we show the work of a student who models the distance of a Ferris wheel rider from the ground as the rider rotates from the bottom of the Ferris wheel (see **fig. 5a**). Previously, this student had explored only a 3 o'clock starting position that involved measuring a directed vertical segment from the circle's center. In the following dialogue excerpt, the student discusses graphing a relationship similar to the sine function and reasons about directional covariation and changing rates of change—precisely the reasoning that is important for a robust understanding of function.

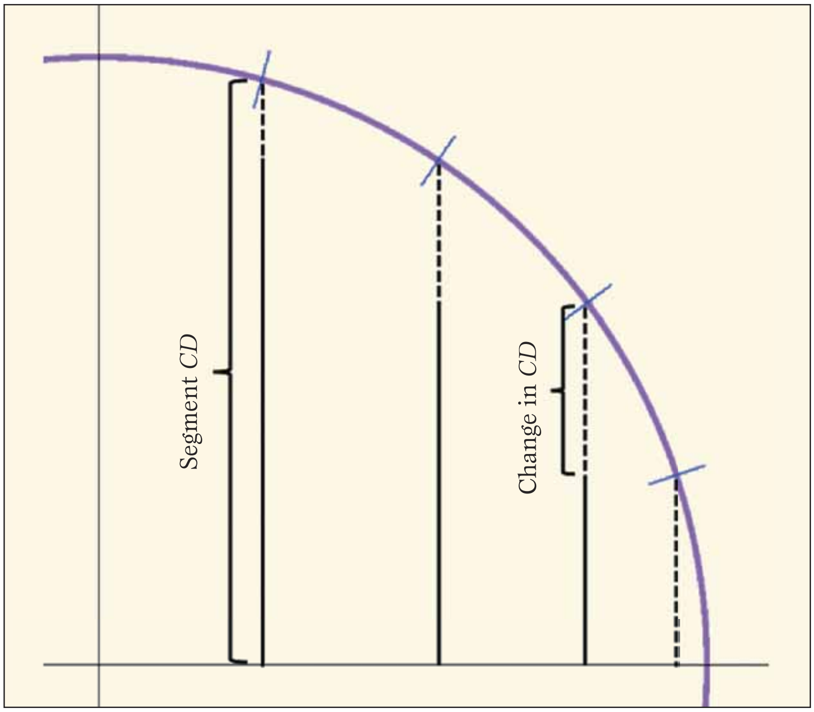


Fig. 3 Students should clearly illustrate changes in vertical distance for equal changes in arc length.

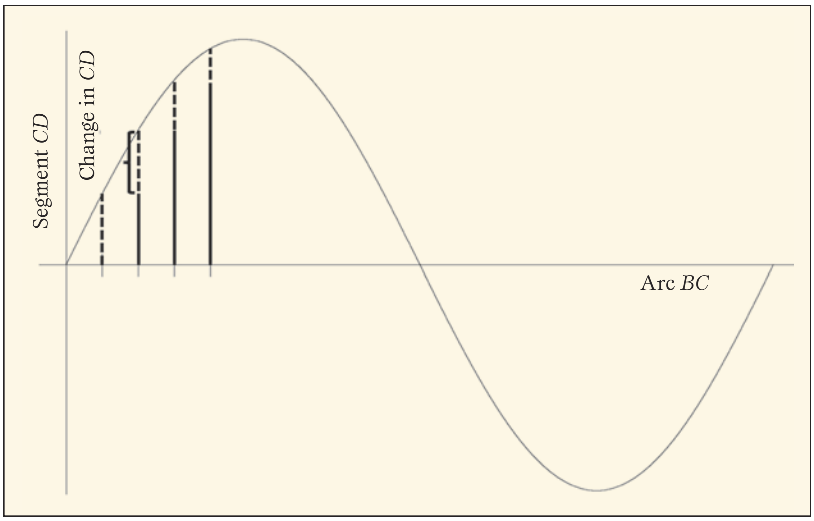


Fig. 4 Students should also clearly illustrate the covariational relationship on the graph.

Student: OK. So a really easy way to do this is divide it up into four quadrants [*divides a circle into four quadrants*]. 'Cause we're here [*pointing to starting position at the bottom of the circle*], for every unit the total distance goes [*tracing successive equal counterclockwise arc lengths*], the vertical distance

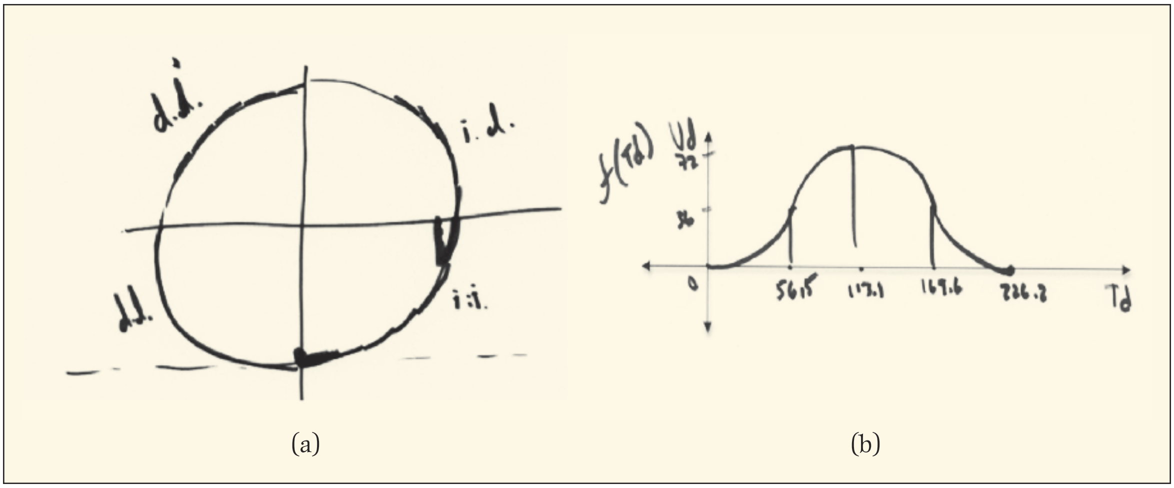


Fig. 5 Modeling the Ferris wheel problem, a student identifies increasing and decreasing segment lengths to help generate a sine function.

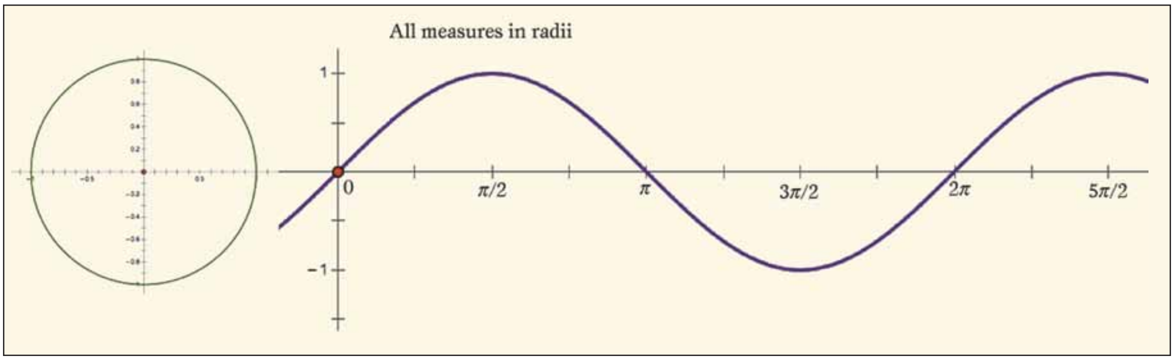


Fig. 6 In both the circle and the graph, the radius is the unit magnitude.

is increasing at an increasing rate [*writing i.i.*] . . . Once she hits 36 feet [*radius of the Ferris wheel*], halfway up, it's still increasing but at a decreasing rate [*tracing successive equal arc lengths, writing i.d.*] . . . [*student continues description for other two quadrants*] . . . So, like, she moves that much there [*tracing an arc length beginning at the starting position*], that much here [*tracing an arc of equal length over the last portion of the path in that quadrant*], the vertical distance there changes by that much [*tracing vertical segment on the vertical diameter*], which is really hard to see with this fat marker. And then the vertical distance here changes by that much [*tracing vertical segment from the starting position of the second arc length*], which is a much bigger change. [*Following this description, the student creates a graph.*]

MEASURING IN RADII AND THE UNIT CIRCLE

For **figure 4** to be a graph of the sine function, the relationship needs to be extended over the domain of all real numbers, and a unit to measure the quantities must be chosen. We can measure each quantity in relation to a standard length unit, such as feet, but, as illustrated by Landers (2013), doing so leads to a graph that conveys only paired numerical values for a circle of specified radius. It is advantageous to have paired values that represent measures on all circles, and so the unit circle and measuring in radii (and radians) provide a critical base for trigonometry.

Many students do not have solid understandings of the unit circle (Moore 2012; Weber 2005). Evidence suggests that their confusion stems from impoverished understandings of radian measure and from conceiving numbers labeled on the circle as unitless. Students label the radius using $r = 1$, but the 1 is not representative of a measure that entails a unit. To students, the radius of a circle is irrelevant in the context of the numbers labeled on the unit circle. If students are to interpret 1 as a measure, a unit must be attached to the number. We do so by introducing the radius length as a unit.

We obtain the unit circle by conceiving a circle's radius as the measuring unit for vertical and horizontal positions (e.g., directed lengths) relative to the circle's center. When students connect the unit circle to measuring in radii, the unit circle represents every circle simultaneously; every circle's radius has a length of 1 when measured in radii (see **fig. 6**). Said in another way, the unit circle represents the result of using any particular circle's radius as a unit magnitude for positions and arcs on that circle. Understanding the unit circle in this way enables students to use the unit circle, radian measures, and trigonometric functions in novel settings.

To illustrate, suppose that the circle in **figure 1** has a 2.7-inch radius. By choosing the inch as the unit of measure, we would graph the directed segment length *CD* as a function of the arc length *BD* using a vertical scale from –2.7 to 2.7 inches (see **fig. 7**). If we choose the radius as the unit of measure for the quantities, we have the graph in $y = \sin(x)$ with vertical scale from –1 radius to 1 radius (see **fig. 6**). The input and output quantities related by the sine function can both be thought of as multiplicative comparisons (i.e., ratios) with a circle's radius.

For example, suppose that we have an angle subtended by an arc of length 5.4 inches on the circle with a radius of 2.7 inches. Evaluating the sine function requires the input to be in radii (or radian) units, so we divide 5.4 by 2.7 to determine that the arc has a measure of 2 radii. Evaluating the sine function at an input of 2 radii yields a value of approximately 0.909, which conveys that the magnitude of the arc terminus's ordinate is 0.909 times (or 90.9%) as large as the radius. Finally, to determine the position in inches, multiply 0.909 by 2.7 to obtain the measure that is 0.909 times as large as the radius (approximately 2.45 in.). This conversion can be generalized to relate a function $f(x) = \sin(x)$, whose output represents a multiplicative relationship with the radius, to a function $g(x) = 2.7 \cdot f(x)$, whose output is in inches.

In the following excerpt of student dialogue, we present a student determining a symbolic form of the relation graphed in **figure 5b**. Specifically, the student represents the Ferris wheel rider's distance from the ground, $f(d)$, as a function of the rider's arc distance traveled in feet, d, by reasoning about the sine function involving input and output values measured relative to the 36-foot radius. The student determines that the sine function input requires the radii measure of an arc from the 3 o'clock position. Because the approximate arc distance in feet between the rider's starting position and the 3 o'clock position is 56.5, the input quantity is represented by $(d - 56.5)/36$.

Student: Vertical distance is equal to *f* of total distance, which is equal to total distance minus 56.5 [*writing*], which will get me there [*pointing to the bottom of the circle*], divided by 36 feet to get me radians. And then I take the sine of that [*referring to the angle measure*] . . . the vertical distance . . . a percentage of the radius length, which I then need to multiply by 36.

Following the dialogue quoted here, the student described adding "one radii [radius] or 36 feet" to determine that

$$f(d) = 36\sin\left(\frac{d - 56.5}{36}\right) + 36.$$

CONNECTING TO RIGHT TRIANGLES

Students who connect angle measure and arcs should understand that an angle measured in right-triangle settings generates a family of circles centered at the angle's vertex. The family of circles includes the circle used to label the angle measure with an arc as well as the circle formed by using the hypotenuse of the right triangle as a radius. Students see that a right triangle is connected to a circle by conceiving of the hypotenuse as a radius. This relationship between the hypotenuse and the radius is critical for giving meaning to the common trigonometry ratios (e.g., SOH-CAH-TOA). In the case of the sine function, we have the ratio of the side opposite from the angle divided by the hypotenuse, which, we can now conclude, represents measuring the side relative to the hypotenuse (or radius).

In both contexts, the sine and cosine functions relate angle measures (with radians being the standardized input unit) and lengths relative to a particular unit (the hypotenuse or radius). Directly tying angle measures to arcs offers a natural connection between the two settings, with the hypotenuse being one choice for the radius of a circle. In addition, connecting the unit circle and right triangles with a focus on quantities and measurement allows students to maintain awareness that trigonometric functions represent dynamic relationships between quantities. Further, evaluating the sine and cosine functions at a specified angle measure, regardless of context, merely represents an instantiation of the covariational relationships defined by the functions.

The following dialogue excerpt reveals a student acknowledging connections between these contexts by identifying that both contexts involve using a length to measure other quantities. This dialogue occurred after the interviewer had asked the student why he had used the hypotenuse of a triangle to form the radius of a circle.

Student: To make it easier to understand. How I was originally taught was just with triangles. Now that we've started using circles, it makes a whole lot more sense . . . because I always just thought the hypotenuse was, you know, just that side of a triangle. You could use the Pythagorean theorem to find out what it was very easily . . . but now [that] I'm looking at it and seeing it's the radius, it makes a lot more sense to be able to find the horizontal and vertical distances according to the radius.

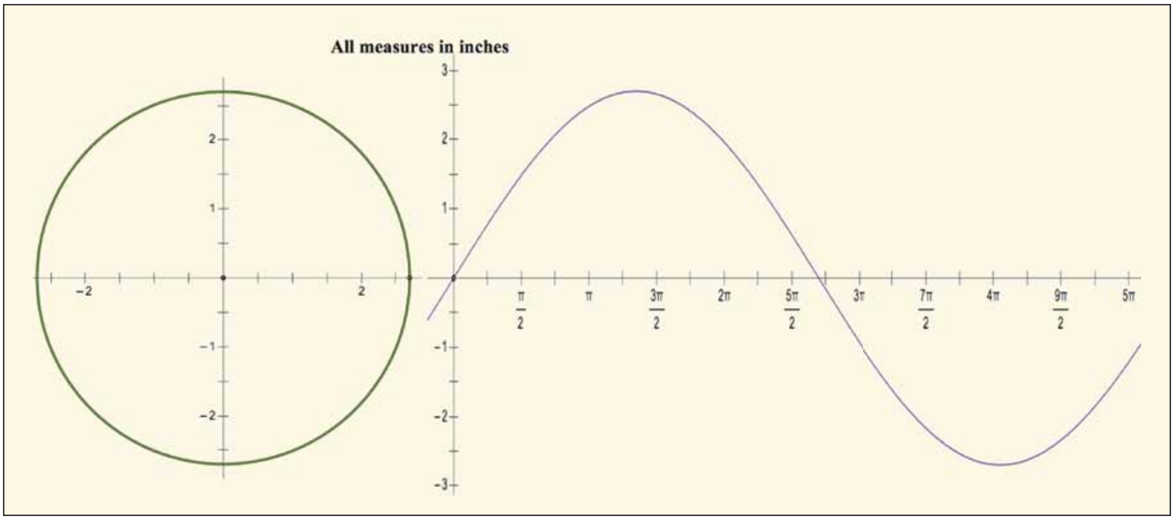


Fig. 7 In both the circle and the graph, the inch is the unit magnitude.

SUPPORTING THE CIRCLE APPROACH

Although the Common Core emphasizes an approach to trigonometric functions that involves quantitative reasoning and circles, research has shown that the teaching and learning of trigonometry frequently lacks a connection to quantities and circles. We argue that an approach to trigonometry that involves quantitative reasoning begins with supporting students' understanding of the quantitative relationships between angle measures and the arc that subtends the angle. With angle measures connected to arcs lengths measured in radii, students can connect the input of trigonometric functions to radian measure. In addition, understanding a circle's radius as the unit of measure is central to a circle approach to trigonometry. Building the unit circle from measurements in radii, students can begin to understand trigonometric functions as covariational relationships applicable to all circles. As Thompson (2008) noted, a triangle context does not naturally enable such a robust covariational and quantitative approach to trigonometric functions.

Despite our focus on circles, we caution readers not to conclude that circle trigonometry is more important than right-triangle trigonometry. Our research suggests that, as teachers consider a coherent introduction to trigonometry, using the unit circle is more suitable for an approach that builds on angle measure and fundamentally involves ideas of measurement, covariation, and equivalence. However, both contexts offer many important applications and uses of trigonometric functions. Thus, the ideas presented here can and should be extended to using trigonometric functions in right-triangle contexts, particularly in exploring similarity and giving meaning to the trigonometric ratios typically introduced within a right-triangle context.

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