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Generalizing is a critical aspect of mathematics learning, with researchers and policy documents highlighting generalizing as a core mathematical practice. It can also be challenging to foster in class settings, and teachers need access to better resources to teach generalizing, including an understanding of effective forms of instruction. This article proposes Classroom Supports for Generalizing (CSGs), investigating how multiple elements—such as tasks, teacher moves, student interactions, and representations—interact to meaningfully foster student generalizing. Drawing on class video data from a middle school teacher and two high school teachers, we present the CSG Framework, which identifies three categories of supports: Interactions for Generalizing, Structures for Generalizing, and Routines for Generalizing.

Keywords: Generalizing; Classroom teaching; Middle school; High school

Generalizing is a central component of mathematics learning. Researchers have argued that generalizing serves as the origin of mathematical ideas (Peirce, 1956; Vygotsky, 1986), and the importance of generalizing is reflected in national standards documents worldwide (e.g., Australian Curriculum, Assessment and Reporting Authority, 2019; Department for Education, 2021; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Secretaría de Educación Pública, 2017). Many curricular materials include activities aimed at fostering generalizing, and research on practices that promote mathematical reasoning highlight generalizing as a key component (e.g., Brodie, 2009; Jeannotte & Kieran, 2017; Schifter & Russell, 2020). For instance, Brodie (2009) emphasized generalizing as a fundamental practice in mathematical reasoning, and Stylianides et al. (2013) identified generalizing and "developing arguments for the truth or falsity of the generalizations" (p. 1465) as the two major activities in reasoning-and-proving exercises.

Definitions of generalizing vary, but most consider it to be an extension of some property to a set of mathematical objects or conditions that is larger than the set of individually verified cases (Carraher et al., 2008). For instance, Radford (2006) defined generalization as the identification of a commonality on the basis of particulars and the subsequent extension to all terms; Harel and Tall (1991) characterized generalization as the process of applying a given argument to a broader context; and Kaput (1999) took generalization to be the shift from considering individual cases to considering the patterns, relationships, and structures across them. As we discuss later, generalizing can also be considered from a whole-class perspective, framed as an activity structured by interaction, task engagement, the use of tools, and class norms.

Despite the importance of generalizing to mathematical reasoning, research shows pervasive student difficulties in creating and understanding correct general statements (e.g., Čadež & Kolar, 2015; English & Warren, 1995; Mason, 1996; Rivera, 2008). These difficulties introduce obstacles to students' success in a variety of mathematical domains (Chazan, 2006; Ellis & Grinstead, 2008; Lockwood & Reed, 2016; Pytlak, 2015). Developing students' capacity to generalize is, therefore, an important component of increasing achievement in mathematics (Vlahović-Štetić et al., 2010). Further, organizing classes in a manner that can support students to take ownership in creating their own generalizations has the potential to center student voices, invite agency, and increase equitable access to mathematics (Gresalfi et al., 2018; Widman et al., 2019).

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Although students' challenges with generalizing are well documented, as a field we know less about how to support generalizing. The bulk of research on teaching for generalizing has occurred in individual or small-group settings, using methods such as clinical interviews or teaching experiments. Less is understood about how productive generalizing can occur in school settings with practicing teachers. The research that does address teachers' abilities to foster generalizing suggests that effectively teaching for generalizing is challenging (e.g., Callejo & Zapatera, 2017; Cockburn, 2012; El Mouhayar & Jurdack, 2013). Teachers need support in learning how to help students generalize, including increased access to research-based findings that build on what we know about students' productive generalizing. In response to these needs, this article investigates the nature of generalizing activity in middle school and high school classes. We draw on data from three teachers' classrooms to investigate the following research question: What are the opportunities for student generalizing, and what factors promote generalizing in the class? Specifically, what are the classroom supports that foster students' generalizing activity?

To answer these questions, we introduce the Classroom Supports for Generalizing (CSG) Framework. This framework considers the interactions that can occur in a classroom setting to identify three types of related supports for generalizing. To demonstrate the application of the CSG Framework, we present two data examples from different classrooms. These examples serve to illustrate how the identified CSGs interact in productive ways to support student generalizing. Additionally, we offer a discussion of how specific classroom supports may be intentionally leveraged to better foster classroom generalizing.

Review of the Literature: Effectively Supporting Generalizing

Many studies within the literature on reasoning and argumentation have examined aspects of generalizing as an integral part of these processes. For instance, Reid (2002), Brodie (2009), Jeannotte and Kieran (2017), and Ellis et al. (2019) all emphasized generalizing as a fundamental practice in mathematical reasoning-and-proving activities. In describing teacher moves that extend student reasoning, Ellis et al. (2019) identified "pressing for generalization" as a powerful move that can effectively extend student reasoning (p. 123). Those investigating reasoning-and-proving activities also included generalizing as a central aspect (e.g., Ellis et al., 2019; Melhuish et al., 2020; Stylianides et al., 2013). In fact, Stylianides (2008) identified a major goal in mathematics education to be that of helping students "make generalizations on the basis of mathematical structures" (p. 10). Similarly, in considering processes of argumentation, Conner et al. (2014) addressed generalizing as part of inductive reasoning. Many of these studies considered how other mathematical practices, such as symbolizing, representing, and communicating, can support generalizing (e.g., Brodie, 2009).

Student Activities for Supporting Generalizing

Researchers investigating the conditions that support generalizing have found that generalizing is effective when students have opportunities to monitor and reflect on their thinking. Synthesizing findings from the literature, we have identified four types of student activity that are useful for promoting generalizing: (a) visualizing, (b) focusing, (c) connecting, and (d) expressing. *Visualizing*, the act of imagining properties beyond what is perceptually available, is an important aspect of generalizing patterns (Amit & Neria, 2008; Yeap & Kaur, 2008). Amit and Neria (2008) and Rivera (2007), for instance, found that visualization played a central role in supporting students' abilities to generalize patterns. Becker and Rivera (2006) addressed the importance of perception, particularly cognitive perception (Rivera & Becker, 2008), which involves recognizing a fact or a property in relation to an object. Visualizing a set of properties beyond what is perceptually available appears to be a key element in generalizing patterns.

Focusing is the act of attending to particular characteristics above others. Specializing on one case in a pattern is a form of focusing (Mason, 1996), and researchers have identified how focusing on particular types of cues can produce meaningful generalizations. For instance, Rivera and Becker (2007) found that students who focused on numerical cues used less sophisticated strategies to generalize compared with those who focused on figural cues.

Connecting concerns students' abilities to identify and reflect on relationships between tasks, representations, or properties. Researchers have emphasized the significance of making connections between representations to promote generalizing (Cooper & Warren, 2008; Dörfler, 2008; Johanning, 2004). One form of connection entails the ability to identify structural similarities across perceptually different objects, which can "quickly facilitate the generalizing process" (Rivera & Becker, 2008, p. 74). Reflecting on connections across representations, on operations, and on generalizations themselves is also an important aspect of fostering generalizing (Amit & Neria, 2008).

Finally, *expressing* (depicting generalizations verbally or in written language) is a key support for generalizing (MacGregor & Stacey, 1993; Rivera & Becker, 2008). When students are encouraged to describe generalizations in words, they can then subsequently develop algebraic generalizations (e.g., Ellis, 2011; Matthews & Ellis, 2018; Stacey & MacGregor, 2001). Consequently, Rivera and Becker (2008) noted that language is an important factor influencing students' abilities to generalize.

Pedagogical Strategies for Supporting Generalizing

Researchers have also investigated pedagogical strategies for supporting generalizing. Teacher moves that have the potential to support generalizing include having students consider big numbers (Zazkis et al., 2008), showing variation across tasks (Mason, 1996), guiding students to reflect on their mathematical operations (Dörfler, 2008; Ellis, 2007a), providing access to representations (Amit & Neria, 2008), emphasizing similarity across contexts (Radford, 2008), and ordering tasks in a progressive sequence (Ellis, 2011; Steele & Johanning, 2004). However, two caveats should be considered in relation to this body of research. The first is that many of these studies were conducted in individual or small-group settings. The second is that although these findings identify specific instructional moves, fewer studies have explicitly considered the role that student–student interaction can play in fostering generalizing. Consequently, the degree to which these findings might translate to whole-class settings with practicing teachers is not well understood.

In response to these limitations, some researchers have considered how teachers can foster generalizing in whole-class settings. Martino and Maher (1999) examined one elementary class and found that teacher moves encouraging students to explain, clarify, or justify their results supported generalizing. Koellner et al. (2008) conducted a study in an eighth-grade class and found that working with an open-ended problem with multiple entry points, having opportunities to visualize a concrete representation, and being able to work collaboratively fostered students' generalizing. The teacher's discursive moves of pushing for algebraic generalizations without supplying answers also contributed to the students' success in generalizing. Mata-Pereira and da Ponte (2017) also identified teacher actions that supported generalizing in a seventh-grade classroom, finding that exploratory questions created opportunities for students to use examples to generalize. Other moves for eliciting generalizing included encouraging students to share their ideas and providing new mathematical information during class discussions. El Mouhayar (2020) found that during triadic dialogue (Lemke, 1990), students and teachers coordinated their gestures, representations, and utterances to promote generalizing.

Others have attended to the role of interaction in fostering generalizing, both in small-group and whole-class settings. For instance, Jurow (2004) studied how students generalized during group interactions in a classroom and found that students coordinated talk, inscriptions, and gestures to generalize using two participation frameworks, linking (proposing comparisons between situations) and conjecturing. Jurow consequently characterized a generalization as "the outcome of activities distributed across people, talk, and inscriptions rather than the product of any individual's thinking" (p. 296). Ellis (2011) also studied the role that interaction played in identifying Generalizing-Promoting Actions, depicting how teachers and students work together to foster generalizing. These actions included publicly generalizing, encouraging generalizing and justifying, and focusing attention on mathematical relationships, among others. Ellis's study was situated in a small-group teaching experiment rather than a whole class setting, but Strachota et al. (2018) later found that the same Generalizing-Promoting Actions were effective in elementary whole-class lessons. We describe Ellis's Generalizing-Promoting Actions Framework in more detail in the Theoretical Perspectives section.

Together, these studies offer insight into the task types, teacher moves, and interactions that can support mathematical generalizing. They offer existence proofs of specific incidents of productive generalizing, accompanied by an analysis of the conditions that afforded those moments. As we have noted, many (but not all) of these studies were set in individual or small-group settings or considered one specific type of support, such as teacher moves or student interactions. Our study builds on these findings to examine how multiple supports—tasks, representations, teacher moves, student interactions, and engagement with artifacts—can interact to foster student generalizing in whole-class settings. In doing so, we identify a comprehensive framework, the CSG Framework, which presents the multifaceted and related supports for generalizing.

Theoretical Perspectives

Scholars have expressed a range of perspectives on generalizing in mathematics. Given our whole-class focus, we are interested in student generalizing both as an individual act and as a socially situated, collective act. To reflect these dual foci, we draw on both cognitive and interactionist perspectives to define and characterize generalizing.

Defining and Situating Generalizing

As described earlier, many definitions of generalizing situate it as an individual, cognitive construct in which students extend a property to a broader set of objects (Carraher et al., 2008; Harel & Tall, 1991; Radford, 2006). Generalizing can also be framed, however, as a collective act distributed across multiple agents set in a specific sociomathematical context (e.g., Ellis, 2011; Jurow, 2004; Tuomi-Gröhn & Engeström, 2003). From this perspective, one can attend to how students' interactions structure generalizing in concert with other factors, such as teacher moves, task engagement, tool use, and classroom norms. Generalizing is taken as a social phenomenon, whereby meaning is negotiated under the ongoing process of interaction (Blumer, 1969; Voigt, 1995).

Following Voigt's (1996) approach to studying classrooms, we draw on the traditions of symbolic interactionism (Bauersfeld, 1995) to consider how students' interactions with tasks, tools, one another, and their teacher all co-contribute to the development of generalizations. From this perspective, mathematical meanings are subject to interpretation and negotiation, as conveyed through interaction. In particular, the meaning of generalizations are negotiated through social interaction (Eckert & Nilsson, 2017). We take the classroom environment as a system, made up of mutually interacting elements, and then consider how the system can support students' mutual construction of meaning as they generalize. This perspective enables the consideration of individual students' reasoning while also providing a way to attend to the interactional processes that supported their reasoning.

Consistent with these traditions, we define *generalizing* as an activity in which learners in specific sociomathematical contexts engage in at least one of the following actions: (a) identifying commonality across cases, or (b) extending reasoning beyond the range in which it originated. Following Ellis (2007b), we call these processes "generalizing," and use the term "generalization" to refer to the outcome of these processes. Our unit of analysis can be one student's actions, the activity of a group of students, or the activity of the entire class. Certainly, an individual learner can generalize, but we may also identify instances of generalizing that are distributed across multiple people (and other elements).

The Relating-Forming-Extending Framework and the Generalizing-Promoting Actions Framework

We drew on two frameworks to help us identify students' generalizations and the classroom supports for generalizing. The first is Ellis et al.'s (2021) Relating-Forming-Extending (RFE) Framework, which distinguishes multiple forms and types of generalizations (Figure 1). *Relating* refers to connecting back to a prior idea or situation, *Forming* is the creation

Figure 1

The Relating-Forming-Extending (RFE) Framework

<table>
<tr><td rowspan="4">RELATING</td><td rowspan="3">Relating Situations: Forming a relation of similarity across contexts, problems, or situations.</td><td>Connecting back: Forming a connection between a current and previous problem or situation.</td></tr>
<tr><td>Analogy invention: Creating a new situation or problem to be similar to the current one.</td></tr>
<tr><td>Recursive embedding: Embedding a previous situation into a new one as a key component of the new task.</td></tr>
<tr><td colspan="2">Relating Ideas or Strategies (Transfer): Influence of a prior context or task is evident in a student's current operating.</td></tr>
<tr><td rowspan="8">FORMING</td><td rowspan="3">Associating Objects: Forming a relation of similarity between two or more present mathematical objects.</td><td>Operative: Associating objects by isolating a similar property or structure.</td></tr>
<tr><td>Figurative: Associating objects by isolating similarity in form.</td></tr>
<tr><td>Activity-based: Relating objects or ideas on the basis of identifying the product of one's activity as similar.</td></tr>
<tr><td colspan="2">Searching for Similarity or Regularity: Searching to find a stable pattern, regularity, or element of similarity across cases, numbers, or figures.</td></tr>
<tr><td colspan="2">Isolating Constancy: Focusing on and isolating a constant feature across varying features without reaching the final stage of fully identifying a regularity, pattern, or relationship across those cases.</td></tr>
<tr><td colspan="2">Establishing a Way of Operating: Establishing a new way of operating that has the potential to be repeated.</td></tr>
<tr><td rowspan="2">Identifying a Regularity: Identifying a regularity or pattern across cases, numbers, or figures.</td><td>Extracted: Extracting regularity across multiple cases.</td></tr>
<tr><td>Anticipated: Describing a predicted stable feature that the student anticipates will hold for future cases.</td></tr>
<tr><td rowspan="5">EXTENDING</td><td colspan="2">Continuing: Continuing an existing pattern or regularity to a new case, instance, situation, or scenario beyond the one in which the generalization was developed.</td></tr>
<tr><td rowspan="2">Operating: Operating on an identified pattern, regularity, or relationship to extend it to a new case, instance, situation, or scenario beyond the one in which the generalization was developed.</td><td>Minor accommodation: Making a minor change to a regularity to extend it to a new case.</td></tr>
<tr><td>Major accommodation: Making a significant change to the structure of a regularity to project it to a far case or to make sense of a new relationship.</td></tr>
<tr><td colspan="2">Transforming: Extending a generalization by changing the generalization to be extended; in contrast to operating, the generalization itself changes in the act of transforming.</td></tr>
<tr><td colspan="2">Removing Particulars: Extending a specific relationship, pattern, or regularity by removing particular details to express the relationship more generally.</td></tr>
</table>

Note. Adapted from Ellis et al. (2021).

Table 1

Generalizing-Promoting Actions

Action	Description
Publicly generalizing	One (i.e., a member of the classroom community) publicly engages in generalizing.
Encouraging generalization	One encourages others to engage in generalizing.
Encouraging sharing of a generalization or idea	One asks or encourages another member to publicly share a generalization, representation, solution, or idea.
Publicly sharing of a generalization or idea	One shares another member's generalization, idea, strategy, or representation with the larger classroom community. This can take the form of revoicing, or publicly validating or rejecting another member's generalization.
Encouraging justification or clarification	One encourages another member to reflect more deeply on a generalization or an idea by requesting an explanation or justification.
Building on an idea or a generalization	One builds on another member's idea, conclusion, or generalization. This can include refining an idea or using it to create a new idea, rule, or representation.
Focusing attention on mathematical relationships	One directs attention to particular aspects of a problem or representation.

Note. Adapted from Ellis (2011).

of a general rule or the identification of sameness, and *Extending* is using a generalization in a new task or context. The RFE Framework distinguishes the three forms and multiple subtypes within each form and is itself an extension of Ellis's (2007b) generalization taxonomy. Both the taxonomy and the framework position generalizing from the student's perspective, considering what students express as relations of similarity, patterns, or rules to be generalizations, regardless of their mathematical correctness. We follow this stance, attempting to determine what students see or express as general, even if it is not the generalization intended by the teacher.

We also drew on Ellis's (2011) Generalizing-Promoting Actions Framework in considering initial classroom supports for generalizing. The Generalizing-Promoting Actions Framework is grounded in the interactionist perspective, in which generalizations are seen as interactively constituted and subject to change through the collective negotiation of meaning (Herbel-Eisenmann, 2003). The framework identifies seven major categories of actions, which we summarize in Table 1. These categories depict interactional moves in the form of public activity (such as generalizing or sharing a generalization or idea), prompts to others (encouraging generalizing, justifying, or sharing), and responses to others (such as building or focusing attention). Several of these categories are similar to the pedagogical strategies identified in the literature, such as guiding students to reflect (Dörfler, 2008), encouraging students to attend to similarity (Radford, 2008), and encouraging students to explain or justify their reasoning (Martino & Maher, 1999; Mata-Pereira & da Ponte, 2017). Because the Generalizing-Promoting Actions Framework emerged from a small-group teaching experiment, it addresses only interpersonal forms of interaction. Given our stance on generalizing as situated in the class environment as a system made up of mutually interacting elements, we anticipated that we would probably identify other supports beyond this framework. In the Data Analysis section, we detail our process of beginning with these two frameworks and then identifying emergent supports that better reflected the class environments in which we collected our data.

Methods

Participants and Data Collection

Our participants were three classroom teachers and their students. We solicited teachers from nearby districts who expressed interest in participating in a study about generalizing. Three middle school teachers and three high school teachers responded, and we asked each potential participant to choose one lesson for observation that included, from their perspective, opportunities to generalize. We conducted one observation in each teacher's classroom with the aim of identifying classes in which we saw evidence of student-centered activity such as group work or active student participation. Of those six classroom observations, three did not show evidence of active student participation. The three remaining teachers were Ms. N, Ms. R, and Mr. J. During our observation of each of those teachers' classrooms, students worked in groups, shared their ideas, and engaged in reasoning through problems. Each of the three teachers indicated that these practices were typical in their classrooms.

The three teachers taught in different schools and each of the schools served diverse student populations. Ms. N was a 3rd-year teacher who taught sixth-grade mathematics in a rural middle school. Ms. N's school is a Title 1 school, which

Table 2

Lessons Observed in Mr. J's, Ms. R's, and Ms. N's Classrooms

Teacher	Class	Lessons	Length of each lesson	Topics
Mr. J	10th-grade algebra 2	3	75 min	• Rules of exponents, *n*th roots • Rules of exponents, rational exponents • Rational exponents, volumes of cubes
Ms. R	Ninth-grade algebra 1	4	80 min	• Linear equations and inequalities • Rational equations and inequalities • Graphing linear inequalities • Solving systems of equations and inequalities
Ms. N	Sixth-grade general math	4	55 min	• Plotting points, properties of the coordinate plane • Properties of the coordinate plane, scaling axes • Determining horizontal and vertical distance • Determining reflections over the *x*- and *y*-axis

means that more than 35% of its participating students come from low-income families. The students in the school were 56% white, 20% Latinx, 13% Black, 6% Asian American, and 5% two or more races. Ms. R was a 6th-year high school teacher in a rural high school. The students were 57% white, 19% Latinx, 15% Black, 5% Asian American, and 4% two or more races. Mr. J. was a 2nd-year high school teacher who taught in an urban Title 1 high school, where the students were 42% Black, 27% white, 24% Latinx, 5% two or more races, and 2% Asian American. Each class was representative of its school demographics.

For each teacher, we conducted classroom observations and follow-up interviews. Before conducting the observations, we asked each teacher to choose a sequence of lessons that offered students opportunities to generalize. We did not give the teachers our definition of generalizing because the observations reported in this article constituted only the initial part of an ongoing project, in which we later enacted a series of professional development opportunities (for more information, see Hamilton et al., 2021; Tasova et al., 2021). Therefore, we sought to first observe each teacher's typical instruction before any professional development sessions. Each teacher taught different mathematics courses: sixth-grade general mathematics (Ms. N), honors algebra 1 (Ms. R), and on-level algebra 2 (Mr. J). Thus, our data came from varied school and mathematics contexts.

We observed four lessons in Ms. N and Ms. R's classrooms, and three in Mr. J's classroom (Table 2). We recorded all lessons with two video cameras, one moving camera aimed at the classroom teacher during whole-class discussion, and one stationary camera aimed at a focus group of three to four students. Each teacher selected the focus group students, choosing students who were talkative and comfortable sharing their ideas. The focus group camera captured their conversations, written work, and gestures. The focus group remained the same throughout the lessons, and the moving camera also captured other student groups during small-group work.

We also conducted a 1-hr, follow-up semistructured, video-recorded interview (Roulston, 2010) with each teacher to investigate their beliefs about mathematics and generalizing and to inquire about how they saw themselves supporting generalizing during the lessons. In the Results section, we primarily draw on the lesson observations rather than the teacher interviews, but at times the teachers' remarks provided additional insight into their pedagogical decisions.

Data Analysis

We transcribed all video data and then enhanced the transcripts to create multimodal artifacts (Bezemer & Mavers, 2011) that included screenshots of the students' work, descriptions of relevant gestures, and descriptions of other features such as intonation, use of tools and representations, and pauses and pacing. We used MaxQDA for all coding, which enabled each project member to independently code and then compare codes across members. In our first round of analysis, we coded all instances of generalizations using Ellis et al.'s (2021) RFE Framework. Each project member independently coded for generalizations, and then the team met to collectively resolve any discrepancies. We found that the extant codes in the RFE Framework captured all instances of generalizations, and thus we did not need to adjust any of the codes or create new codes.

We then moved onto the second round of coding to identify classroom supports for generalizing. First, we needed to determine whether an action counted as supporting generalizing. We began with our instances of generalizations, coded from Round 1, and then worked backward to examine the actions leading up to each generalization. We coded an action as supporting a generalization if (a) the generalization occurred as a direct response to the action, (b) the generalization reflected a new idea introduced by an action, or (c) we could identify a conceptual chain linking the ideas introduced by an action and the generalization that followed it. For example, in one episode, students drew a fractal tree, relating the number of branches to the number of years the tree grew, and developed a generalization that $(\sqrt[n]{x})^m = x^{m/n}$. Before this, they had developed expressions such as $x^{1/n}$, but had not extended to a non-unit fraction exponent. The teacher asked the students to explain what the expression $x^{1/n}$ meant, a form of Questioning, and then continued Building on their answer by reframing the 1 in $1/n$ as indicating going back in time to Year 1. Later the students used the same reasoning introduced by the teacher to realize that they could think about going back in time to Year 1 and then forward m years, resulting in the expression $x^{m/n}$. Thus, we coded the teacher's Questioning and Building moves as supporting the generalization because we could see their influence on the students' reasoning that led to the generalization. In other instances, we could not discern any effect of the teacher's Questioning move toward a generalization and, in fact, students may not have produced a generalization at all. In those instances, we would not have coded the move as supporting a generalization. We acknowledge that the term "supporting" may connote a sense of causality, but affirming causality is not possible given our study design. Rather, we use the term "supporting" to convey that we interpreted an action as one that we conjecture could have supported the resulting generalization.

The Round 2 coding began with Ellis's (2011) Generalizing-Promoting Actions Framework providing the initial set of codes (Table 1). We found all the Generalizing-Promoting Action codes useful for making sense of our data, but each code was too general to account for the nuances that emerged. For instance, consider the code Publicly Sharing. The Generalizing-Promoting Actions Framework defined Publicly Sharing as sharing another's generalization, idea, strategy, or representation. We found it useful to distinguish the forms of sharing into separate codes, and we also found other forms of sharing not in the Generalizing-Promoting Actions Framework, such as sharing one's own or another's numerical or mathematics fact answer, one's own or another's problem-solving strategy, one's own justification, and so forth. We also found other interactions that were not present in the Generalizing-Promoting Actions Framework, such as different forms of Responding, as well as broader structures of repeated actions that played out in each classroom. We ultimately described these with respect to the various ways that the teachers structured whole-class discussion or enacted Routines for Generalizing. The Generalizing-Promoting Actions Framework did not have these codes, probably because it was a consequence of an analysis of a small-group teaching experiment.

During this round of analysis, we first coded one lesson from each teacher collectively, discussing actions we observed from the Generalizing-Promoting Actions Framework and other actions not captured by the framework. We collectively developed a coding scheme through this process, relying on the constant comparative method (Strauss & Corbin, 1990) to iteratively develop and revise existing codes. Once we had a coding scheme in place, each of the six members of the research team coded each lesson independently. We went through the coding process lesson by lesson, each time meeting as a group to collectively compare and reconcile codes. If, during the individual coding process, any member did not think an existing code captured what they saw in the data, then that member used the parent code (e.g., Encouraging Sharing). We then discussed, as a group, whether a new code was warranted and, if so, we defined the code in a memo. At times, we found that no new code or subcode appropriately fit the data excerpt; in those instances, we applied only the parent code. The process of comparing and reconciling resulted in many iterative changes to our coding scheme; in total, we had 11 different versions of the coding scheme before the final version. We reached the final version once our collective comparing and reconciling sessions did not yield any new codes or any changes to existing codes.

During this round, we also decided that a particular action could be coded with more than one code. Furthermore, the grain size for applying our codes varied, and could be as small as a line of transcript or as large as an extended class discussion, depending on the code. For example, one of our subcodes under Questioning is asking for justification, similar to Ellis's (2011) Encouraging Justification or Clarification. The grain size for that code could be as small as a line of transcript when a teacher asks, "Can you explain why that makes sense?" In contrast, as we will describe in the Results section, other codes address broader structures or routines that could conceivably span the length of an extended class discussion and hundreds of lines of transcript.

In a third and final round of coding, a subset of three rotating members of the research team independently recoded the entire data corpus. By rotating, we ensured that no two lessons were coded by the same three members of the group. The entire research team then met to collaboratively resolve any discrepancies. Our decision to collaboratively resolve discrepancies and strengthen our common understanding of our codes is consistent with Syed and Nelson's (2015) and Morse's (1997) recommendations for establishing rigor in qualitative research. They argue for the value of having research teams develop and refine coding systems collaboratively to afford multiple nuanced readings of the data, with later coding decisions informed by previously analyzed transcripts in an interpretive process. This process enabled the development of a well-argued, robustly generated set of codes and definitions.

Results: Classroom Supports for Generalizing

We found three major categories of CSGs: (a) Interactions for Generalizing, (b) Structures for Generalizing, and (c) Routines for Generalizing. In the following sections, we first identify and describe the three categories and then demonstrate how the major categories of CSGs interact using two data examples.

Interactions, Structures, and Routines for Generalizing

Figure 2 depicts the three CSG categories, which operate at three different grain sizes. *Interactions for Generalizing* refer to interactional moves, that is, the statements, questions, responses, or ideas that people, tasks, or representations that members can introduce into the class conversation. *Structures for Generalizing* are the moves (including task development) that teachers employ to structure students' activity in a manner designed to lead to a generalization. *Routines for Generalizing* are particular types of interactional routines, namely, the patterned and recurrent ways that instruction unfolds in a classroom.[1] Figure 2 also shows the total number of times we assigned each code across the entire data corpus of the three teachers' classrooms. The number of codes for Interactions is greater than the number of codes for Structures and Routines because whereas Interactions can be short statements with codes applying to individual lines of transcript, Structures and Routines apply to broader sections of the entire lesson. Thus, we may code a large chunk of the lesson with one Routine for Generalizing code and then apply dozens of Interaction codes within that chunk. Additionally, the totals

Figure 2

The Three Categories of Interactions, Structures, and Routines for Generalizing

INTERACTIONS FOR GENERALIZING

Encouraging Generalizing (164)
- Encouraging relating (14)
- Encouraging forming (66)
- Encouraging extending (73)
- Encouraging reflection generalization (8)

Sharing (547)

Sharing one's own:
- Generalization (107)
- Problem-solving strategy (88)
- Numerical or mathematical fact answer (174)
- Justification or explanation (69)

Sharing another's:
- Generalization (8)
- Problem-solving strategy (7)
- Numerical or mathematical fact answer (9)

Encouraging Sharing (114)
- Of a generalization (19)
- Of a problem-solving strategy (44)
- Of a numerical or mathematical fact answer (26)

Questioning (996)
- Asking for *confirmation* about the correctness of a problem-solving strategy, generalization, or calculation (204)
- Asking for *guidance* for how to approach a mathematical problem or develop a problem-solving strategy (33)
- Asking for an *explanation* of a mathematical idea, strategy, or relationship (184)
- Asking for a mathematical *justification* or *proof* (10)
- Asking for *clarification* of the meaning of one's strategy, numerical answer, or idea (177)
- Asking for a numerical, mathematical fact, or procedural *answer* (382)

Responding (393)
- *Affirming* the correctness of a mathematical generalization, strategy, or solution (238)
- *Correcting* one's generalization, strategy, or solution (52)
- *Invalidating* the correctness of a mathematical generalization, strategy, or solution (45)
- *Building* on another's generalization, strategy, or solution (41)
- *Boundary clarification* for cases in which a generalization does or does not hold (16)

Telling (194)
- *Telling* students a mathematical fact, procedure, strategy, or idea

STRUCTURES FOR GENERALIZING
- **Structuring by Task**: Presenting task(s) designed to lead to a generalization (9)
- **Structuring by Action**: Drawing attention to mathematical structure (24)
- **Modeling:** Illustrating the process of developing a generalization (7)

ROUTINES FOR GENERALIZING
- **Collecting and Sharing** mathematical strategies or answers for group discussion (5)
- Examining sameness across **Multiple Examples to Form a Rule** (8)
- **Continuing and Justifying**: Asking students to use a newly developed generalization and repeatedly justify it (4)
- **Assess** students' mathematical strategies, provide **Feedback**, ask for the **Next Move** (18)

[1] We use the phrase "Routines for Generalizing" to refer to a set of stable actions a teacher repeats across multiple lessons. The relationship between this type of routine and Leinhardt and colleagues' notion of an instructional routines (e.g., Leinhardt & Greeno, 1986; Leinhardt & Steele, 2005) is left for future work.

for the different categories of Interactions, for example 164 for the Encouraging Generalizing category, are sometimes greater than the sum of the individual subcodes within that category (in this case, the sum of the Encouraging Generalizing codes is 161). This is because in some instances we applied the parent code but not a subcode.

Interactions for Generalizing

Interactions for Generalizing are in-the-moment teacher or student moves that occur throughout the course of a whole-class or small-group conversation. Consistent with symbolic interactionism, Interactions for Generalizing capture how all members of the classroom community come to negotiate meaning by initiating questions, responding to one another, sharing strategies or ideas, or encouraging others to share, generalize, or justify. In addition, a person's use of task prompts, problems, or representations can also constitute an interactional move if such a use plays an in-the-moment role of fostering generalizing. Within this category, classroom members can engage in the actions of Encouraging Generalizing, Sharing, Encouraging Sharing, Questioning, Responding, or Telling. *Encouraging Generalizing* refers to cases in which one explicitly asks or encourages another member to make a generalization. Relying on the RFE Framework (Ellis et al., 2021), we distinguished the different types of generalizations one can encourage another to make, namely, relating, forming, extending, or the creation of a reflection generalization.

We also found that Sharing one's own ideas, whether prompted or not, can foster generalizing. This was true when teachers and students shared their own generalizations, problem-solving strategies, justifications or explanations, or numerical answers or mathematical facts. Furthermore, members also could foster generalizing by sharing someone else's generalization, problem-solving strategy, or numerical answer or mathematical fact. Encouraging Sharing also fostered generalizing, and this occurred in the form of encouraging another member to share their generalization, problem-solving strategy, or numerical answer or mathematical fact.

The last three sets of interactions for generalizing concern classroom members' engagement with one another through Questioning, Responding, and Telling. Both students and teachers can engage in Questioning. Some moves are more commonly made by students, such as asking for confirmation that an idea, generalization, problem-solving strategy, or mathematical answer is viable. We also found that students often asked for guidance, for example, by asking for help when they were unsure about how to begin a task or when they needed support in developing or using a particular strategy. Other moves were more commonly made by teachers, such as asking for a mathematical fact, procedure, or numerical answer; asking for an explanation about how one solved a problem, thought about a mathematical idea, or reached a solution; or asking for a justification, which could be a request for an informal argument or a formal proof. Asking for clarification was a move that both students and teachers commonly made; this occurred when they were unsure about the meaning of another's problem-solving strategy, numerical answer, solution, or idea, and attempted to clarify that meaning. Although some moves did occur more often with teachers or with students, we found that none of the questioning moves were made exclusively by students or by teachers.

We also found that both teachers and students engaged in Responding and Telling actions, although teachers more often made these moves. Class members responded to one another by affirming or invalidating the correctness of another's idea, strategy, generalization, or solution, and sometimes by correcting another's contribution. Classroom members would build on one another's ideas, either by refining the mathematical idea or by using it to create a new idea, mathematical rule, or representation. The last responding move is boundary clarification, in which one responds to another's generalization by identifying cases for which it did or did not hold. Finally, we found that Telling could also promote generalizing. Here we follow the description from Lobato et al. (2005) of *Telling* as initiating actions that introduce new mathematical ideas into a classroom conversation. These actions can be direct, such as introducing a fact or strategy or directing students to approach a task in a particular way, but they can also be indirect, such as encouraging students to attend to a certain set of features that would reveal a particular strategy or idea. As we will show in the data episodes, these interactions formed meaningful and intertwined supports for generalizing.

Structures for Generalizing

If Interactions for Generalizing are localized and predominantly spontaneous, Structures for Generalizing address both the localized aspects of instruction and the aspects that are more systematic and intentional. Structuring actions are the moves a classroom member employs to explicitly or implicitly shape others' activity in a manner designed to lead to a generalization. We found three types of Structures, which we discuss here in order from least to most directed. The first, *Structuring by Task*, concerns the instances in which a teacher implements a task or a set of tasks with the aim of building to a particular generalization. *Structuring by Action* refers to cases in which a teacher focuses students' attention in a manner that supports the development of a generalization. This can include explicitly drawing students' attention to sameness across problem types or ideas, or organizing a series of representations in a manner that highlights a generalizable feature. This action is broader than the Encouraging Generalizing interactions in that it encompasses more than just a direct request for a generalization and instead concerns a series of actions one makes when structuring a classroom conversation. The third structuring action, Modeling, addresses cases in which classroom members model the process of developing a generalization for other members of the community; as such, it can be quite directive in nature. By Modeling, one makes explicit and public their process when forming a generalization; thus, our use of the term "modeling" here is different from mathematical modeling (e.g., Lesh & Doerr, 2003).[2] We found instances of both students and teachers modeling the process of generalizing.

Routines for Generalizing

Routines for Generalizing depict the patterned recurrent ways that instruction can unfold in a classroom (Horn & Little, 2010). They have a stable schematic core with a more fluid shell, allowing for variable responses to in-the-moment demands (Rösken et al., 2008). *Collecting and Sharing* entails collecting a range of student strategies to share for whole-class discussion and to serve as a source for forming a generalization. Multiple Examples to Form a Rule entails publicly sharing multiple cases, strategies, representations, or other examples, directing students' attention to sameness across the examples, and then encouraging the development of a general rule as a representation of that sameness. Continuing and Justifying occurs once a rule had been developed. Teachers can invite students to use the newly developed generalization for a new task and to justify the validity of the generalization in each case. The final routine, Assess, Feedback, Next Move, occurred during small-group or individual work. In this routine, the teacher visits a student or a group, assesses their progress toward a generalization, provides feedback and guidance on the basis of this progress, and then suggests a specific next step to achieve. Each of the routines we identified appeared repeatedly in one teacher's classroom but seldom appeared in the other classrooms, with the exception of Continuing and Justifying. This suggests that many routines may be somewhat teacher specific.

In the next section, we present two data episodes to illustrate the manner in which multiple CSGs operate together. The first episode is from Ms. N's sixth-grade classroom (ages 11–12) and occurs within the Routine Multiple Examples to Form a Rule. The second episode is from Mr. J's 10th-grade algebra 2 class (ages 15–16), and is an example of the Routine Assess, Feedback, Next Move. In both episodes, the more fine-grained Interactions for Generalizing occurred within a set of larger Structures and Routines for Generalizing, highlighting how smaller grained CSGs may be more or less effective depending on the larger context in which they occur.

Data Episodes

Multiple Examples to Form a Rule: Horizontal Distance

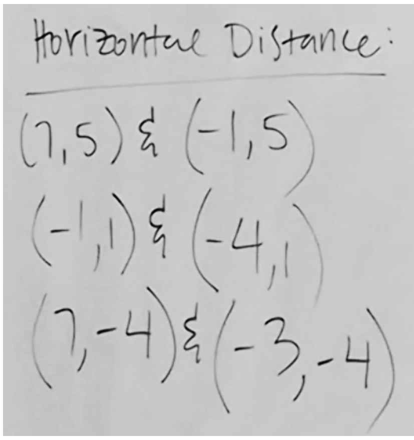
When Ms. N was enacting Multiple Examples to Form a Rule, she would highlight several cases of the same phenomenon before directing the students to identify sameness to develop a mathematical rule. This routine supported both Structuring by Action and Modeling, as well as a number of Interactions for Generalizing. For the remainder of this section, we foreground the interactions that emerged during a whole-class conversation that built toward the final generalization for determining the horizontal distance between two points. Ms. N restricted her instruction of horizontal distance to determining the distance between two points, (x_1, y_1) and (x_2, y_2), that shared the same y-coordinates. This was consistent with the Georgia *Mathematics Standards*, particularly Standard 6.PAR.8.3, "Include use of coordinates and absolute value to find distances between points with the same x-coordinate or the same y-coordinate" (Georgia Department of Education, 2021, p. 89). Ms. N referred to this as "vertical distance" and "horizontal distance." We recognize that one can determine the horizontal distance between any two points by taking the absolute value of the difference $x_1 – x_2$, regardless of their y-coordinates, but that is not how Ms. N interpreted the idea of horizontal distance during this lesson.

In launching the routine, Ms. N projected a coordinate plane on the board and placed a magnetic dart at the point (7, 5). She then asked a student to place a second dart a horizontal distance of eight units from the first dart. The student placed the dart at the point (−1, 5), and Ms. N told the students to write down the two ordered pairs. She then repeated this process two more times, placing a dart at (−1, 1) and asking a student to place a second dart at a horizontal distance three units away, and then placing a dart at (7, −4) and asking a student to place the second dart a horizontal distance of 10 units away. At this point, Ms. N engaged in Structuring by Action: She wrote the three pairs of ordered pairs together on the board, lining them up to highlight the fact that the y-values of each pair of ordered pairs was the same (Figure 3). The representation itself played the role of Encouraging Generalizing (forming) by directing students' attention to the structure of each pair.

[2] Here we use the term "modeling" to refer to the process of demonstrating behavior or thinking for others—for instance, as parents do when they put their clothes in the laundry basket every night, or demonstrate how they express feelings of frustration. This type of modeling is not necessarily intentional. Children learn how to behave by observing their parents' behavior, even when parents do not intend to teach their children to behave in particular ways.

Figure 3

Ms. N's Representation of Three Pairs of Ordered Pairs



In Table A1, we give each classroom member's utterance with the accompanying CSG it represents. The excerpt begins with Ms. N Encouraging Generalizing (forming) by asking the students to consider what the first pair of ordered pairs had in common. The next line is Ari Sharing her own generalization that it has the same *y*-coordinate. Ms. N repeats the same question for the other two pairs of ordered pairs, with the students noting that each pair has the same *y*-coordinate, and Ms. N concludes by Sharing her own generalization, stating, "So that is going to be a pattern that you will always notice whenever we are talking about horizontal distance between two points."

Ms. N uses Structuring by Action throughout the exchange by drawing students' attention to sameness across the three pairs of ordered pairs. She does so not only through the directed conversation, but also by gesturing to the *y*-values on the board and by underlining those values as the students spoke. She then further engages in Encouraging Generalizing (forming) by asking the class, "Is it possible that I could look at these ordered pairs and without even plotting them, know the distance between them?" Jonah proposes that they could simply take the sum of the absolute value of the *x*-values of each pair of ordered pairs to find the difference:

Jonah: You just need to add them together. You can get how many things you go over. Because the top (*Pointing to [7, 5] and [−1, 5]*) like if you, you add them together, but you get rid of the negative sign, it equals eight. Second (*Pointing to [−1, 1] and [−4, 1]*) you move five.

Ms. N: Okay. So be careful with, with saying add them together. I think I know what you mean. But be careful with say add them.

With his proposal, Jonah is Sharing his own generalization. In response to Ms. N's caution, he subsequently clarifies that he meant to say absolute value. Ms. N reacts by Responding with a boundary clarification by asking the students to consider the ordered pairs (−1, 1) and (−4, 1). The class members determine, by counting the number of units between the two points, that Jonah's generalization works for the first and third pairs of ordered pairs but not for the middle pair.

The students and Ms. N then begin with Jonah's initial generalization that one adds the absolute value of the *x*-coordinates for any two points and proceeded to collectively build on one another's statements to develop a modified generalization (see Table A2). In particular, Riley contributes by Responding (building), proposing a modification that one can subtract the absolute values for the pair of points that had *x*-values with the same sign. Ms. N does not take up this modification. In her postlesson interview, she shared that she did not do so because her students had not yet learned arithmetic with negative numbers, so she feared Riley's contribution might be confusing to some of the other class members. Instead, Ms. N redirects the conversation toward the idea of distance, emphasizing that one could just count to determine distances:

Ms. N: So, if I'm finding the distance, Jonah, between a positive number and a negative number, you're right, I am going to need to know their absolute value so that I can combine them. But if they're already on the same side of zero, I can literally just do what? I can count one, two, I can just count the distance, right?

The interactions of Questioning, Sharing, and Responding are supportive of generalizing in part because they occur within the broader Structuring by Action move and the Multiple Examples to Form a Rule routine for generalizing. These broader structures support the development of the final generalization for determining distance as the students and the teacher mutually negotiate meanings for absolute value and addition and subtraction of x-values.

Assess, Feedback, Next Move: Fractal Trees

In contrast to Ms. N's Multiple Examples to Form a Rule Routine for Generalizing, Mr. J's Routine for Generalizing was Assess, Feedback, Next Move. Within this routine, Mr. J would offer opportunities for students to explain and justify their thinking, both to him and to one another, and he made meaningful efforts to understand his students' reasoning before building on that reasoning and encouraging them to further generalize. Within the routine, Mr. J also used a Structure for Generalizing, but instead of Structuring by Action, he used Structuring by Task, employing task features to lead to a generalization.

To illustrate, we draw on two episodes from Mr. J's 10th-grade algebra 2 class, during which the students drew fractal trees to represent exponential growth, and then developed generalizations about rational exponents. During this class, Mr. J encouraged students to work together in groups on a series of questions about exponential growth (Figure 4). Mr. J shared in his interview that he created the task in Figure 4 to support the development of the generalization that $(\sqrt[n]{x})^m = x^{m/n}$. He first created the series of questions for the fractal tree (Question 1) to help the students reason with the values they had drawn, leading them to the generalization that for the number of base branches b, in Year n the tree will have b^n branches. Questions 2–9 then structured increasingly general relationships for different base and exponent values.

We consider the interactions of two groups. The first group, Eli, Kiara, and Aria, created a doubling fractal tree, with two new branches each year. The second group, with Ted and Cameron, created a tripling fractal tree (Figure 5). The students' fractal tree drawings themselves served the role of Structures for Generalizing, because the representations drew attention to the repeated structure of doubling or tripling.

Figure 4

Fractal Tree Exponential Growth Tasks

1. How many new branches grew in Year 1?
   Year 2?
   Year 3?
   Year 4?
   Year 5?
   Year 6?
   Year 10?
   Year 25?

2. If you had 3125 new branches in year 5, then how many did you have in year 1? How could you write this with rational exponents?

3. If you had 1,000,000,000 new branches in year 9, then how many would you have in year 1? How could you write this with rational exponents?

4. If you had 1,000,000,000 new branches in year 9, then how many would you have in year 10? How could you write this with rational exponents?

5. If you had 512 new branches in year 3, then how many would you have in year 2? How could you write this with rational exponents?

6. If you had 512 new branches in year 3, then how many would you have in year 5? How could you write this with rational exponents?

7. If you had m new branches in year 2, then how many would you have in year 1? How could you write this with rational exponents?

8. If you had z new branches in year 7, then how many would you have in year 2? How could you write this with rational exponents?

9. If you had x new branches in year n, then how many would you have in year 1? How could you write this with rational exponents? How many would you have in year m?

Figure 5

Ted and Cameron's Fractal Tree

Eli, Kiara, and Aria. When Mr. J visits the first group, they have successfully identified the number of branches up through Year 6, but are stuck on Years 10 and 25. The exchange in Table A3 entails a number of Questioning and Responding moves. Unlike the tables in the prior excerpt, Table A3 sometimes depicts several exchanges that all reflect a single CSG. Aria begins the exchange by Questioning, asking for guidance about how to approach the problem. Mr. J responds with his own Questioning move, asking for an explanation of her strategy for finding branch values up to Year 10. In doing so, he provides the space for Aria to revisit her actions in creating the number of branches for each year by doubling. As she reflects on her prior activity, Aria then appears to have a realization, stating, "Oh! It's 16." Through Sharing her own justification or explanation, she appears to realize that that if the number of branches for Year 6 is 64, then she needs to multiply 64 by 16 to get the number of branches for Year 10. Implicit in this realization may be the understanding that 16 is the result of doubling four times, which is an action that Aria has just completed to find the other branch numbers. Here we see the influence of Structuring by Task, in that providing students with repeated opportunities to double reinforces the connection between the number of branches and the number of times one must multiply by 2.

Mr. J then builds on Aria's realization by pointing out the difference from Year 6 to Year 10 being 4 years, and then by using his fingers as a way of Modeling the activity of doubling four times in a row, another way in which he used a Structure for Generalizing. Mr. J then uses Questioning (asking for a mathematical fact), pushing the students to express 16 with "exponential terms," hoping that they would say that 16 is 2^4: "How could we use what we're talking about with exponential terms? Can you use an exponential term to write 16?" When Aria responds with four squared, Mr. J. subtly invalidates the correctness of that answer by suggesting that they were "almost there." He then leaves them to grapple with the task on their own, completing the Routine for Generalizing of Assess, Feedback, Next Move.

We do not have access to the group's conversation when Mr. J is not present, but by the time he returns, Aria has written "2^4" on her paper. She has also written each branch number using exponential notation (i.e., 2^6 for Year 6, 2^{10} for Year 10, and so forth) and then is able to solve the number of branches for Year 25 by calculating 2^{25}, enacting a generalization that the total number of branches in Year n will be 2^n. We suspect that having the opportunity to negotiate her meanings for doubling may have supported Aria in beginning to connect the activity of doubling with the number of times she has doubled. In addition to Structuring by Task, Mr. J's encouragement to think about this action in exponential notation, combined with Modeling the connection between repeated doubling and shifting from one year to the next, may have supported the students to ultimately generalize the year as the exponent.

Ted and Cameron. Ted and Cameron are comfortable with rational exponents of the form $1/b$; for example, they can express the answer from Question 2 as $3{,}125^{1/5}$. However, they experience challenges using nonunit rational exponents, that is, rational exponents of the form a/b, $a > 1$. When the students encounter Question 5, they first take the cubed root of 512, yielding eight branches, and then square the result to get 64 branches. The students express this as $512^{1/3}$, even though they know that they have to take that outcome and square it to get the final answer. Ted and Cameron can imagine two separate processes, first taking a root and then raising the result to a power, but they do not yet have a way to use one rational exponent to express both processes.

When the students reach Question 9, they are able to write the expression $x^{1/n}$ in response to the first part, "If you had x number of new branches in Year n, then how many did you have in Year 1?" However, they are stuck on the second question, "How many would you have in Year m?" Before this question, they had successfully written one expression with a nonunit rational exponent, $z^{2/7}$, but they were unsure of their answer and did not understand the rational exponent to simultaneously represent the seventh root of z and raising z to the second power. At this point, Mr. J visits the group and asks them to explain their thinking (see Table A4).

After Assessing the students' progress through Questioning (asking for an explanation), Mr. J then provides an implicit form of Feedback by repeatedly building on the students' expressions with rational exponents. He does so by positioning the denominator of the exponent, n, as what takes them back to the number of branches in Year 1. Mr. J then continues Questioning, asking for an explanation for the students' response to Question 8, $z^{2/7}$. He emphasizes the denominator, 7, as "you went back 7 years," but then asks about the meaning of the numerator, 2. In Sharing his own justification or explanation, Cameron then takes up Mr. J's language by describing the numerator as "going back [from Year 1] to Year 2." In Responding, Mr. J emphasizes this again by pointing to the numerator and noting, "Once you got back, you started going forward again," and "You started growing again." Mr. J understands a fractional exponent m/n as a representation of a two-step process, in which one takes the nth root, and then raises the resulting outcome to the mth power; the nth root "takes you back" to Year 1, and then the mth power "takes you forward" to the number of branches in Year m. Mr. J also engages in another Structure for Generalizing, Modeling the process of developing a generalization. Having communicated this image to the students, Mr. J then provides the students with the Next Move, Encouraging Generalizing by asking them to extend their reasoning from Question 8 to the final Question 9.

In response to Mr. J's suggestion, Ted says to Cameron, "Oh, I guess we could just put m divided by n. Instead of 1," providing evidence that he had renegotiated his meaning for the fractional exponent. In doing so, he is Sharing his own generalization, extending the reasoning they had used to write the rational exponent 2/7 to the more general form of expressing the number of new branches in Year m if the tree had x branches in Year n. Cameron, however, is unsure whether Ted's idea was correct. She suggests that they first think about the expression with numbers rather than variables, Sharing her own strategy (see Table A5).

Cameron and Ted engage in several Sharing, Questioning, and Responding moves, deciding to treat Year n as two and x as four. Cameron is comfortable using a unit fraction, such as $1/n$, to think about finding the number of branches in Year 1, but struggles to then connect that action to the subsequent action of raising the resulting value to the power of m to determine the number branches in Year m. Thus, Cameron pauses after stating 4 to the power of 1/2. Ted responds by reminding her that she then has to think about Year m, which helps Cameron then attempt to mentally consolidate two processes: Finding the number of branches by taking the square root (or nth root), which she could express as a rational exponent $1/n$, and then raising the result to the power m. In the process of mutually negotiating meanings for the fractional exponents and the processes they represented, Ted helps Cameron think about how to express that consolidation algebraically as m/n.

Throughout the episode, the classroom members' moves of building on one another's ideas support Cameron's final generalization. Mr. J builds on the students' written work to offer a description of what he sees their expression $z^{2/7}$ to mean. Cameron then takes up his language in her exchange with Mr. J, but only after Mr. J challenges the students to extend that reasoning to the final question are Cameron and Ted able to generalize not just Mr. J's language about going back or moving forward, but also the idea that a two-step process could be represented by one rational exponent. In their exchange with each other, we also see instances of Cameron and Ted building on each other's ideas and expressions to reach the final generalization. We see that the Responding actions, such as affirming, correcting, and building, constitute meaningful supports toward generalization in the context of the task structure; notably, these are actions that students make, as well as teachers.

In both groups from Mr J.'s class, Sharing is also an important interactional move. Through Sharing their thinking Aria realizes that she needed to multiply 64 by 16 to get the number of branches for Year 10, and Cameron takes up Mr. J's idea that the numerator of a rational exponent returns you to that year. Mr. J's instructional routine of Assess, Feedback, Next Move is one that offers many opportunities for Sharing, both with him when he asks the students to explain their thinking, and with one another when he gives the students a Next Move and the space to explore it on their own. These Sharing moves may not have been as powerful had they occurred within a different Routine for Generalizing.

The Interaction of Multiple Elements as Classroom Supports for Generalizing

The previous episodes illustrated how multiple elements can be CSGs, including not only teacher moves, but also students' interactions, tasks, representations, and gestures. We saw students' interactions operating as CSGs when Riley built on Jonah's generalization to account for the sign of the x-values in the ordered pairs, when Aria's Sharing of her own justification helped her see that she needed to multiply 64 by 16 to get the number of branches for Year 10, and when Ted and Cameron acted collectively in Sharing a strategy, correcting it, and building on it to write a general expression with a fractional exponent of m/n. We saw gestures operating as CSGs when Ms. N underlined the y-values in the ordered pairs on the board to encourage forming, and when Mr. J used his fingers as a way of Modeling the activity of doubling four times in a row. Representations operated as CSGs when the lining up of the three ordered pairs in Ms. N's class encouraged forming, and when the fractal trees in Mr. J's class served as a Structure for Generalizing by drawing attention to the action of repeated multiplication. We also saw the tasks themselves serving as Structures for Generalizing, such as when Mr. J designed a series of questions providing repeated opportunities to make connections between the number of branches and the number of times one multiplies.

The CSG Framework also helps us make sense of how the three categories interact to support generalizing. For instance, Ms. N's routine of Multiple Examples to Form a Rule supported Structures for Generalizing, such as Structuring by Action. Ms. N engaged in Structuring by Action by writing three pairs of ordered pairs on the board, by gesturing to the *y*-values, and by underlining those values. These actions complemented the Interactions for Generalizing that occurred, such as Encouraging Generalizing by asking what the pairs of ordered pairs had and did not have in common, and Questioning by asking students to state the *y*-coordinates. The students' actions of Sharing that the *y*-coordinates were the same further served to emphasize the general rule that points on the same horizontal line will have the same *y*-coordinate. Occurring in a different context, absent any structuring actions or the presentation of multiple examples, moves such as Questioning, Sharing, and Responding may not necessarily be effective.

Interactions for Generalizing do not have to occur within a Structure or Routine to be productive, but we observed that many of the most powerful instances of generalizing occurred when multiple supports interacted together. Moreover, considering the opposite direction, Structures and Routines for Generalizing will be productive only to the extent to which students are willing to negotiate meaning in interaction. For instance, Mr. J's use of the Assess, Feedback, Next Move routine was effective because his students openly shared their thinking, felt comfortable asking questions, and were willing to attempt further reasoning on their own once Mr. J stepped away. The routine may not have effectively supported generalizing if Mr. J's students had not embraced the norms of independent and collective reasoning.

Discussion

Fostering mathematical generalizing is challenging for teachers (Callejo & Zapatera, 2017; Cockburn, 2012; El Mouhayar & Jurdack, 2013), and our participants were no exception. But at the same time, the CSG Framework enabled us to identify instances of teachers and students engaging in activities together in ways that supported powerful moments of generalizing. The CSG Framework affirms a number of supports for generalizing identified elsewhere in the literature. Supports such as Encouraging Generalizing, Structuring by Action or Task, and the routine Multiple Examples to Form a Rule align with previous findings that teachers can foster generalizing by showing variation across tasks and ordering tasks in a progressive sequence, and by helping students make connections between representations and identify structural similarities across objects or contexts (Amit & Neria, 2008; Cooper & Warren, 2008; Ellis, 2011; Mason, 1996; Radford, 2008; Rivera & Becker, 2008; Steele & Johanning, 2004). Several of our Sharing, Encouraging Sharing, and Questioning interactions, including moves such as sharing one's own or another's generalization, encouraging sharing of a generalization or strategy, asking for an explanation, asking for a mathematical justification or proof, and asking for a numerical or mathematical fact answer are consistent with the research showing that encouraging students to describe their ideas and generalizations with language can support the subsequent ability to develop algebraic generalizations (Ellis, 2011; Mata-Pereira & da Ponte, 2017; Matthews & Ellis, 2018; Rivera & Becker, 2008; Stacey & MacGregor, 2001). Researchers have also found that encouraging students to reflect on their operations and create algebraic generalizations can be effective (Amit & Neria, 2008; Dörfler, 2008; Koellner et al., 2008); these moves are consistent with our interactions, such as Encouraging Generalizing and asking for a justification or proof. Even Telling has been shown to support student generalizing (Mata-Pereira & da Ponte, 2017).

The CSG Framework also extends the literature in two ways. First, it introduces previously unidentified supports for generalizing. These include Questioning and Responding Interactions for Generalizing, such as asking for confirmation, asking for guidance, asking for clarification, asking for a numerical answer or mathematical fact, affirming correctness, invalidating, correcting, building, and clarifying a generalization's boundary. Notably, many of these Interactions were ones that the students engaged in as much as their teachers. This represents a departure from existing studies, the majority of which considered instructional moves to support generalizing without explicitly addressing the students' roles. The CSG Framework also introduces Modeling as a powerful Structure for Generalizing, as well as four Routines for Generalizing.

Second, the CSG Framework is the first treatment of classroom generalizing that coordinates different grain sizes, from the in-the-moment Interactions for Generalizing to the broader Structures and Routines for Generalizing, accounting for how these different types of supports can work together to support generalizing within the whole-class environment. In this manner, it extends Ellis's (2011) Generalizing-Promoting Actions Framework by not only determining additional aspects of that framework's generalizing-promoting actions, but also by considering and analyzing supports for generalizing within varied settings, including whole-class, small-group, and one-on-one interactions.

The Role of Symbolic Interactionism

A teacher's goals, actions, and interpretations can form interactions that guide the negotiation of meaning in the class (Eckert & Nilsson, 2017). From an interactionist perspective, meaning is produced through a process of negotiation, and researchers can observe teachers' and students' actions, what those actions indicate, and how they are interpreted by others (Blumer, 1969). This perspective provided a way for us to determine whether an action constituted a CSG. In some instances, a teacher's discursive move was not taken up by students, even when it was a direct encouragement to generalize, and so the move did not constitute a CSG. In other instances, however, we were able to observe how students negotiated meaning through interaction—with one another, with their teacher, with task features, or with a representation—and in those negotiations, develop generalizations. As Voigt (1996) argued, symbolic interactionism offers a way to attend to both social processes, such as the establishment and negotiation of sociomathematical norms (e.g., Yackel & Rasmussen, 2002), as well as individual meaning making.

This perspective provided a productive lens for characterizing CSGs by observing how multiple aspects of the class environment contributed to the negotiation of meaning in ways that led to generalizing. Even Routines for Generalizing, which at first glance may seem to be exclusively a depiction of teachers' actions, can be understood as a consequence of coordination among students, teachers, and the mutual and intersecting goals driving their behavior. Routines develop and solidify in part because of how teachers perceive their effectiveness unfolding over time in relation to their experiences with their students. As we saw with both the Multiple Examples to Form a Rule routine with Ms. N and the Assess, Feedback, Next Move routine with Mr. J, not just the routine itself, but also the interactions that occurred within each routine, and the students' taking up of the teachers' and one another's moves, mutually interacted to support generalizing.

The CSG Framework grew out of a stance that considered the class to be a system of mutually interacting elements (Voigt, 1995). Thus, it affords a system-based perspective on supporting generalizing. How any given Interaction may or may not support generalizing must be understood in the context of the other Interactions, Structures, and Routines in which it occurs. Any one move's effectiveness depends not only on the immediate context in which it takes place, but also on the larger system of exchanges that forms the sociomathematical milieu of the class.

Future Work

The CSG Framework offers potential for both understanding how generalizing can occur in the classroom and for supporting teachers in making shifts to their instruction to more effectively foster generalizing. At the conclusion of the study reported in this article, we continued our collaboration with the participating teachers and shared our findings about the actions we observed in their classrooms that supported their students' generalizing. Naming the Routines and Structures, in particular, could help teachers reflect on the aspects of their practice that they can leverage to help their students generalize and to contemplate what they could do differently. By examining the types of interactions that were prevalent in each teacher's classroom, our participants began to think about small shifts they could make in their Questioning and Responding, for example, shifting from asking for answers to asking for explanations and justifications. This preliminary work suggests that teacher educators and teachers could use the CSG Framework as a tool for making sense of whole-class interactions and for reflecting on their practice for supporting generalizing.

The current study also suggests several avenues for future research. First, we do not assume that the framework is exhaustive. We developed the framework from observing three teachers in three content areas, and more work is needed to determine how our findings might translate to other classes and particularly other grade levels. Such work may identify additional Interactions, Structures, and Routines that prove productive in supporting generalizing. In particular, we suspect that other Routines for Generalizing could be found. All three of our participants habitually enacted routines specific to them, and other teachers are likely to have different routines that are also useful for promoting generalizing.

In addition, our study investigated the state of classroom generalizing from the perspective of trying to understand the supports that are already present in teachers' classrooms. Much remains to be understood about how to purposefully design classroom experiences to foster students' generalizing. We envision a productive line of inquiry to be one that builds on our current findings to create, implement, and study interventions that capitalize on the ways in which teachers teach for generalizing. We find it encouraging that teachers can and do meaningfully support generalizing in their classrooms. The CSG Framework provides one lens for understanding how those processes occur and it may be a tool that teachers could find useful in designing and implementing lessons for fostering generalizing.

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APPENDIX

Whole-Class and Small-Group Interactions for Generalizing

Table A1

Identifying Commonalities in the Ordered Pairs

	Utterance	CSG
Ms. N:	Who can tell me what looked at these two ordered pairs to start (*points to the first pair*). What do they have in common? What are these ordered pairs have in common?	Encouraging Generalizing (encouraging forming)
Ari:	They both have the same *y*-axis coordinate?	Sharing (one's own generalization)
Ms. N:	*y*-coordinate. Good. What is their *y*-coordinate?	Questioning (asking for a mathematical fact answer)
Ari:	5.	Sharing (one's own mathematical fact answer)
Ms. N:	5 (*underlines the y-values*). All right, what do these two points (*points to the next pair*) have in common? Rayna?	Encouraging Generalizing (encouraging forming)
Rayna:	They have the same *y*-coordinate?	Sharing (one's own generalization)
Ms. N:	What is the *y*-coordinate?	Questioning (asking for a mathematical fact answer)
Rayna:	1.	Sharing (one's own mathematical fact answer)
Ms. N:	They both have a 1 in common in the *y*-coordinate place (*underlines the* y-*values*) and what do these two points have in common (*points to the last pair*)? Wesley.	Encouraging Generalizing (encouraging forming)
Wesley:	They both have the same *y*-axis coordinate which is −4.	Sharing (one's own generalization)
Ms. N:	Perfect (*underlines the* y-*values*). So, what do they *not* have in common? What are they not sharing?	Encouraging Generalizing (encouraging forming)
Parker:	*x*-coordinate.	Sharing (one's own generalization)
Ms. N:	Their *x*-coordinates, right? So that is going to be a pattern that you will always notice whenever we are talking about horizontal distance between two points.	Sharing (one's own generalization)

Table A2

Transforming Jonah's Generalization

	Utterance	CSG
Ms. N:	So what, what happened with your theory? I like the theory, it's almost there. But we need to tweak it a little bit going.	Encouraging Generalizing (encouraging forming)
Jonah:	I think we are going negatives to positives. I think it only works with positive negative, positive positive.	Sharing (one's own generalization)
Ms. N:	And try them if my two coordinates are not the same sign, you mean?	Questioning (asking for clarification)
Jonah:	You change the negative, you just kind of do the opposite.	Sharing (one's own generalization)
Ms. N:	Okay, cool, can be something to add to our rule.	Responding (affirming the generalization)
Riley:	This one, like go, go ones that he's talking about adding. They start with the positive number. And when we, with this (*points to [−1, 1] and [−4, 1]*), and it starts with negative number, you can subtract it from before, and equals 3.	Responding (building on the generalization)
Jonah:	Yeah, that's what I mean by like negative, negative.	Responding (affirming)
Ms. N:	Okay, so in general, what am I looking for? Absolute value is asking us for a, what do we say? What kind of measurement?	Questioning (asking for a mathematical fact answer)
Robin:	Distance.	Sharing (one's own mathematical fact answer)
Ms. N:	A distance. So, in general, this is always going to be true. What am I looking for between the two points that aren't the same?	Questioning (asking for a mathematical fact answer)
Quinn:	Positive number.	Sharing (one's own mathematical fact answer)
Ms. N:	I'm looking for, the word you just said—	Questioning (asking for a mathematical fact answer)
Riley:	(*Interrupts*) Distance.	Sharing (one's own mathematical fact answer)
Ms. N:	I'm looking for the distance between them, right? So, if I'm finding the distance, Jonah, between a positive number and a negative number, you're right, I am going to need to know their absolute value so that I can combine them. But if they're already on the same side of zero, I can literally just do what I can count one, two, I can just count the distance, right? Like I know from −1 to −4. It's how far—	Telling (a general mathematical strategy)
Jonah:	(*Interrupts*) I think that only works when they are both at opposite sides.	Responding (building on another's generalization)
		Sharing (one's own generalization)
Ms. N:	Yeah, I think that's true if they don't have the same sign, I like your strategy.	Responding (affirming)

Table A3

Determining the Number of Branches for Year 25

	Utterance	CSG
Aria:	Look Mr. J. We got in kind of a jam.	Questioning (asking for guidance for how to approach the problem)
Mr. J:	You got in a jam?	
Aria:	We're trying to figure out . . .	
Mr. J:	Does it have peanut butter with it too? Because peanut butter can be sticky and then you're sticky and in a jam.	
Aria:	We wrote it out like this and we . . . (*inaudible*; *shows her paper to Mr. J indicating the branch values up through Year 10*)	
Mr. J:	Sweet. How'd you do that?	Questioning (asking for explanation of their strategy)
Aria:	But when we did it, we multiplied each one by 2.	Sharing (one's own justification or explanation)
Mr. J:	You kept multiplying by 2.	
Aria:	Yeah, but when we tried to do, um, we tried to make it . . . oh! It's 16.	
Mr. J:	64, and you did 16 because you had to cover 4 years, right?	Responding (building on their justification)
Mr. J:	So, how many times was that, to multiply together?	Questioning (asking for a numerical answer)
Aria:	We was thinking, we was thinking 8 at first. But then, um, it gave us Year 9, so we was like, we had to go bigger and do 16.	Sharing (one's own justification or explanation)
Mr. J:	Uh huh. And 16, that would be 4 years I'm thinking, because (*holds up four fingers and uses them to count*) that's 2 (*folds down one finger*), times 2 (*folds down the second finger*), times 2 (*folds down the third finger*), times 2 (*folds down the fourth finger*).	Responding (building on their justification) Modeling (the process of developing a generalization)
Mr. J:	How could we use what we're talking about with exponential terms? Can you use an exponential term to write 16?	Questioning (asking for a mathematical fact)
Kiara:	Yeah, you could write 2 to the . . . (*trails off*).	
Mr. J:	Think about it, okay?	
Aria:	Could do 4, 4 squared?	Questioning (asking for confirmation of correctness)
Mr. J:	Y'all are on it. Y'all are almost there. I'm going to give it to you. I'm going to turn it over to you, because you're there (*walks away from the group*).	Responding (invalidating the correctness)

Table A4

Explaining Rational Exponents as "Going Back" and "Going Forward"

	Utterance	CSG
Mr. J:	Alright. And tell me what that is (*pointing to the students' expression* $x^{1/n}$). What's that mean?	Questioning (asking for an explanation of a mathematical relationship)
Ted:	What I thought was, like up here (*points to Question 2*), it's like the fifth root of that, and you just put 1 divided by the number of roots. So wouldn't it be the same there (*points to their expression* $x^{1/n}$)?	Sharing (one's own justification or explanation) Questioning (asking for confirmation)
Mr. J:	So this took you back to Year 1, didn't it?	Responding (building on the strategy) Questioning (asking for a mathematical fact)
Ted:	Yeah.	Responding (affirming)
Mr. J:	However many you had in some year, you go that many, oh, or, yeah, you go so many years back. I think I see what y'all are talking about there. Alright, so let's shift into the . . . these, and this is what you did here too? This took you back to Year 1 (*points to the students' answer of* $m^{1/2}$ *for Question 7*)?	Responding (building on the strategy) Questioning (asking for clarification of the strategy)
Ted and Cameron:	Yeah.	Responding (affirming)
Mr. J:	And this took you back to Year 1 (*pointing to the students' expression in Question 8,* $z^{2/7}$*, which they had written as a guess*)? But you did something else here. But you did something else here. You went back 7 years to Year 1. And then what's, what's that about (*points to the numerator of the fractional exponent, 2*)?	Responding (building on the strategy) Questioning (asking for an explanation of a mathematical relationship)
Cameron:	We had to. . .	Questioning (asking for an explanation of a mathematical relationship)
Mr. J:	Where'd that 2 come from?	
Cameron:	Because you're going back to Year 2. So, you had to multiply it by Year 2.	Sharing (one's own justification or explanation)
Mr. J:	Okay. So, you go all the way back to Year 1. And then you have, you can pop back out from there.	Responding (building on the justification)
Cameron:	Yeah.	Responding (affirming)
Mr. J:	Alright. Let's look at seven (*reads the question*). If you had *m* number of new branches in Year 2 (*points to their answer*), how many did you have in . . . I see. I see. I'm following your thinking now (*points to their answer for Question 8,* $z^{2/7}$). So, this took you back—all of these took you back. And now this one took you back and once you got back, you started going forward again.	Responding (building on the strategy) Modeling (the process of developing a generalization)
Ted:	Yeah.	Responding (affirming)
Mr. J:	Once you came back to the roots, you started growing again. Alright, so how do you start growing again here (*points to Question 9*; *students remain silent*)? I'm going to check back in, okay (*leaves the group*)?	Modeling (the process of developing a generalization) Encouraging Generalizing (encouraging extending)

Table A5

Confirming Ted's Generalization

	Utterance	CSG
Cameron:	So, let's pretend that these all have numbers. So—	Sharing (one's own strategy)
Ted:	Shouldn't we just use, like, 2 or something?	Responding (building on the strategy)
Cameron:	Yeah, that's what I'm doing. So, if you had *x* number of new branches in Year *n*. So, Year *n* is 2, and maybe *x* is 4. Then how many would you have in Year 1?	Sharing (one's own strategy)
Ted:	So *n* is 2, *x* is 4?	Questioning (asking for confirmation of correctness)
Cameron:	Yeah. So, so then you would just do 4 to the power of 1/2. And that half would be . . . well, no, because we're finding . . . (*trails off*)	Responding (affirming correctness) Sharing (one's own strategy)
Ted:	You still have to find *m*. I want to finish this question.	Responding (correcting the strategy)
Cameron:	Yeah, me too. I feel like it'd be easier if just like one thing was not an exponent.	Sharing (one's own strategy)
Ted:	Yeah. Like if *m* was like an actual number.	Responding (affirming)
Cameron:	Yeah, then that would be . . . so, maybe we have to go back to 2 which is 1/2, but then we have to multiply by *m*. So maybe it's . . . can we do that? Can you do, like, 1/2 times *m*? Is that like an acceptable . . . (*trails off*)	Questioning (asking for confirmation of correctness of strategy)
Ted:	Couldn't you just write *m* divided by 2 instead?	Responding (correcting the strategy)
Cameron:	That, that's so true. Yeah! So, then it would be, that would make it *x* equals *m* over *n* (*writes* $x^{m/n}$), which is what we found. So, I think that answer makes the most sense.	Responding (building on the strategy)