The Parametric Nature of Two Students' Covariational Reasoning

Teo Paoletti
Montclair State Univeristy

Kevin C. Moore
University of Georgia

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The parametric nature of two students' covariational reasoning

Teo Paoletti[a,*], Kevin C. Moore[b]

[a] Mathematical Sciences Department, Montclair State University, 1 Normal Ave., Montclair, NJ 07043, United States
[b] Department of Mathematics and Science Education, University of Georgia, 105 Aderhold Hall, Athens, GA 30602, United States

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ABSTRACT

Researchers have argued that covariational reasoning is foundational for learning a variety of mathematics topics. We extend prior research by examining two students' covariational reasoning with attention to the extent they became consciously aware of the parametric nature of their reasoning. We first describe our theoretical background including different conceptions of covariation researchers have found useful when characterizing student reasoning. We then present two students' activities during a teaching experiment in which they constructed and reasoned about covarying quantities. We highlight aspects of the students' reasoning that we conjectured created an intellectual need that resulted in their constructing a parameter quantity or attribute, a need we explored in closing teaching episodes. We discuss implications of these results for perspectives on covariational reasoning, students' understandings of graphs and parametric functions, and areas of future research.

Parametrically defined functions appear in U.S. and international pre-calculus and calculus curricula (Bergeron, 2015), and a sophisticated understanding of parametrically defined functions is critical to understanding ideas in advanced mathematics. For instance, reasoning parametrically is central to interpreting solutions of systems of differential equations (Trigueros, 2001, 2004). Although researchers have recently examined students' parametric reasoning and understandings of parametric functions (Keene, 2007; Stalvey & Vidakovic, 2015; Trigueros, 2001), these examinations were in the context of calculus and differential equations courses in which students were formally introduced to parametric functions. Further, reasoning that is parametric in nature can serve as a foundational basis for students' reasoning about relationships between covarying quantities (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017); there is, however, limited research explicitly examining the parametric nature of students' covariational reasoning.

We contribute to the aforementioned bodies of literature by describing two students' covariational reasoning with attention to the parametric nature of their reasoning. In doing so, we provide empirical examples of students reasoning parametrically (as defined below) and becoming consciously aware of the parametric nature of their reasoning. Relatedly, we highlight how the students' reasoning raised what Harel (2007) defined as an *intellectual need*: "a behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate" (p. 13).

In what follows, we first summarize the extant research on students' understanding of parametric functions. We then provide our theoretical background including perspectives on covariation that informed this study. We describe the methods we used to investigate and develop models of the students' covariational reasoning. We then focus on the students' actions during teaching experiment episodes in order to characterize the parametric nature of the students' reasoning as inferred during ongoing and retrospective analyses. We close with a discussion of our findings and suggestions for future research.

* Corresponding author.
E-mail address: paolettit@mail.montclair.edu (T. Paoletti).

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1. Literature related to students' parametric reasoning

Relative to examinations of students' function understandings, there is a dearth of literature examining students' parametric reasoning and understanding of parametric functions. Trigueros (2001, 2004) described three stages of students' understandings of parametric functions —intra-parametric, interparametric, and trans-parametric—in an applied dynamical systems course. At the intra-parametric stage, students can interpret the components of the parametric representation ($f(t)$, $g(t)$) in terms of the two functions f and g and the parameter t. However, these students experience difficulties relating those functions to the geometric and graphical representations of the parametric function and interpreting a graphical representation in which the parametric variable is not explicitly represented. Students at the interparametric stage can interpret a graphical representation in which the parametric variable is not represented, but they experience difficulties interpreting parametric functions as vectors. Students at the trans-parametric level, the highest level, do not experience any of the aforementioned difficulties. Trigueros (2004) classified 20 of the 37 students in her study at the intra-parametric stage, highlighting that a large proportion of students in advanced mathematics courses do not maintain sophisticated understandings of parametrically defined functions *after* formal instruction on the topic.

Keene (2007) examined students' dynamic reasoning, defined as "developing and using conceptualizations about time as a dynamic parameter that [an individual] implicitly or explicitly coordinates with other quantities to understand and solve problems" (2007, p. 231), in a differential equations course. Keene's characterizations of how students reasoned with time can inform studies on the parametric nature of students' reasoning. For example, Keene identified that students reasoned dynamically (i.e., with time) as they considered how two different attributes of a physical situation changed. In addition, Keene (2007) noted students often incorporated time in their descriptions of relationships represented by graphs in which time was not explicitly represented (i.e., position-velocity graphs representing a mass on a spring), indicating that reasoning dynamically may be natural for some students in particular contexts.

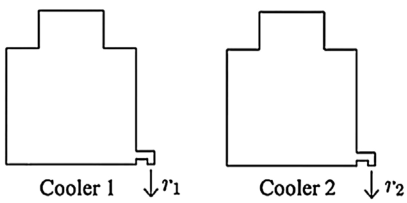
Also relevant to the present study, Stalvey and Vidakovic (2015) presented a genetic decomposition—a description of how an individual might construct an understanding of a specific mathematical concept—for parametric functions. In the first step, a student conceives two functions f and g, both with an argument of t, and possibly coordinates the outputs of the functions by comparing changes in $f(t)$ with changes in $g(t)$ over small intervals of t. In the second step, the student coordinates his understanding from step one with his understanding of Cartesian coordinates to represent coordinate points $(x, y) = (f(t), g(t))$ while understanding that the point in the coordinate system traces out a curve defined by $(f(t), g(t))$. The last two steps involve students re-parameterizing the functions to remove the parameter t and understanding the relationship between x and y as invariant.

In order to investigate their genetic decomposition with 15 Calculus II students, Stalvey and Vidakovic (2015) designed a series of tasks involving water draining from two identical bottles, each with a constant drain rate but with a different drain rate with respect to the other bottle (Fig. 1). Reflecting the first step in their genetic decomposition, the researchers initially asked students to create height-time and volume-time graphs for each bottle. The researchers then asked the students to represent the relationship between volume and height without explicitly representing time. Finally, they asked the students to represent the direction each curve traced out as the situation progressed. Stalvey and Vidakovic (2015) noted that a majority of the students in their study had difficulties graphically representing the relationship between volume and height. The authors concluded that conceiving an invariant relationship between two varying quantities, each of which are dependent on a third quantity, is a non-trivial task.

We draw attention to Stalvey and Vidakovic (2015) including time as an explicit variable in their first two prompts. Explaining their task design, and noting that the relationship between volume and height in each situation is identical, the authors included time explicitly, "so that it encompasses the idea of an oriented curve, which is central to the concept of parametric function" (p. 203). We note, however, that the authors did not provide data describing how the students' activities graphing volume-time and height-time relationships supported the students in constructing or orienting their volume-height graphs. Hence, Stalvey and Vidakovic did not provide evidence regarding the extent students need to explicitly coordinate or represent two quantities with respect to a third quantity before coordinating or representing the two quantities together (i.e., step one and two in their genetic decomposition).

2. Theoretical background

An increasing number of researchers have made contributions to the literature base on students' quantitative and covariational



Assume that Coolers 1 and 2 above are the same size. Imagine that they are full of water and being emptied at constant rates r_1 and r_2, respectively. Assume that $|r_1| < |r_2|$.

Fig. 1. The Task, reproduced from Stalvey and Vidakovic (2015, p. 196).

reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Larsen, & Lesh, 2003; Castillo-Garsow, 2012; Confrey & Smith, 1995; Ellis, 2007; Ellis, Ozgur, Kulow, Williams, & Amidon, 2015; Johnson, 2012; Thompson, 1994a, 1994b). These contributions have been with respect to students' understandings of various topical areas (e.g., function classes, rate of change, and the fundamental theorem of calculus) and to their enactment of important mental processes (e.g., generalizing, modeling, and problem solving). Although maintaining the common intention of understanding students' covariational reasoning, researchers' treatments of covariation are varied.

2.1. Confrey and Smith's characterization of covariation

Providing a contrast to a rule-based correspondence approach to function, Confrey and Smith (1991, 1994, 1995) described a covariational approach to function in which they viewed functions as, "The covariation between quantities. As one quantity changes in a predictable or recognizable pattern, the other also changes, typically in a differing pattern" (1991, p. 57). Confrey and Smith indicated that a student could describe a functional relationship by describing changes in consecutive x-values and then the changes in corresponding y-values, each represented in a table of values. Confrey and Smith (1995) further argued that students' covariational approach provided them with a productive understanding of exponential functions and the authors claimed that the function concept in general would be better understood from a covariational perspective. This claim has been supported by recent research which has shown that elementary, middle, and high school students can develop sophisticated understandings of functions by reasoning covariational about values presented in tables (Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015; Ellis et al., 2015; Ferrari-Escolá, Martinez-Sierra, & Méndez-Guevara, 2016).

2.2. Thompson and colleagues' description of covariation

Differing from Confrey and Smith's approach to covariation, Thompson and Saldanha (Saldanha & Thompson, 1998; Thompson, 2011) defined covariation in terms of coordinating changes in two continuous *magnitudes*, thus not constraining covariation to the availability of specified numerical values. Saldanha and Thompson (1998) stated, "Our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 298). Such an understanding entails the cognitive coupling of two quantities to form a multiplicative object; when defining a multiplicative object, Saldanha and Thompson relied on Piaget's notion of "and" as a multiplicative operator (Inhelder & Piaget, 1958), which refers to a cognitive uniting of two attributes to create an object "that is, simultaneously, one and the other" (Thompson & Carlson, 2017, p. 433).[1] With respect to covariation, Saldanha and Thompson (1998) described, "As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (p. 298).

Extending this description, Thompson (2011) characterized such an understanding in which an individual conceives of a quantity's value, x, varying over conceptual (i.e., imagined) time, t. Clarifying the notion of conceptual time, Thompson and Carlson (2017) stated:

> We distinguish between experiential time and conceptual time as follows: Experiential time is the experience of time passing, whereas conceptual time is an image of measured duration. We say *image* of measured duration to dispel interpretations that someone must think he is actually timing an event. Rather, we are speaking of someone imagining a quantity as having different values at different moments, and envisioning that those moments happen continuously and rhythmically. (emphasis in original, p. 444–445).

Hence, when an individual has conceived of a situation as constituted by quantities and conceptual time, they can imagine and represent these quantities continuously changing independent of the direct or in-flow sensorimotor experience of the situation. Thompson and Carlson (2017) elaborated that conceiving of a situation as entailing conceptual time involves a conceptual shift that allows the student to disembed time, and thus their image of a varying quantity is "new because time [is] no longer tacit in it, and they also [have] a quantity that is tantamount to conceptual time (imagined measured duration)" (p. 432). By holding in mind a quantity varying with respect to conceptual time, Thompson (2011) argued that an individual could then conceive of covering the domain of t-values using intervals of size ε (i.e., $(x(t), x(t+\varepsilon))$), and consider the variation of x within these intervals (i.e., considering $x(t_\varepsilon)$ as the continuum of x-values $(x(t), x(t+\varepsilon))$). Thompson (2011) then described a conception of two quantities values covarying as $(x(t_\varepsilon), y(t_\varepsilon))$, which he intended "to represent conceiving of a multiplicative object—an object that is produced by uniting in mind two or more quantities simultaneously" (p. 47).

Drawing on Thompson and Saldanha's (Saldanha & Thompson, 1998; Thompson, 2011) perspective of covariation, Moore and Thompson (Moore, 2016; Moore & Thompson, 2015) defined *emergent shape thinking* as a student conceiving graphs in terms of an emergent, progressive trace constituted by covarying magnitudes. We use Fig. 2 to convey Moore and Thompson's notion of emergent shape thinking. The reader should interpret Fig. 2 as instantiations of an emergent image of a trace representing liquid height and liquid volume in a bottle covarying as liquid is poured into the bottle; the reader should imagine an emergent trace being created between the instantiations in Fig. 2. Adopting Thompson's (2011) notation, the student understands height, $h(t_\varepsilon)$, and volume, $v(t_\varepsilon)$, both increase as conceptual time, t, increases. A student with such an image of a graph understands that the magnitude of the

[1] For more on Thompson and colleagues' use of multiplicative object, see Thompson and Carlson (2017) and Thompson, Hatfield, Yoon, Joshua, and Byerley (in press).

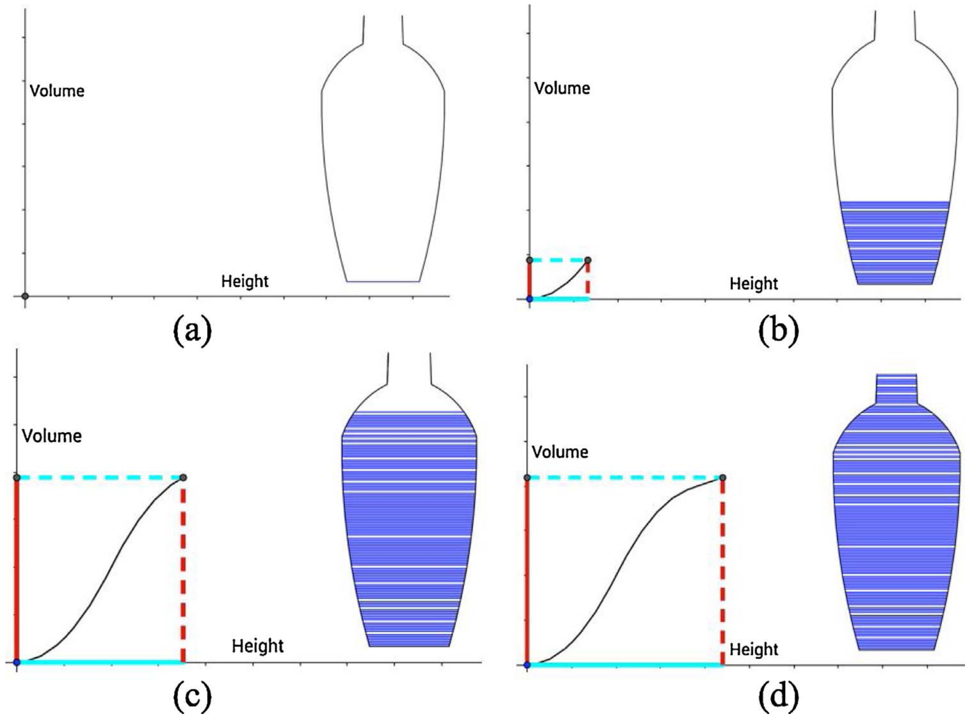


Fig. 2. Four instantiations of an emergent conception of height (horizontal, blue) and volume (vertical, red) of liquid in a bottle covarying as liquid is poured into a bottle. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

horizontal blue segment represents the height of liquid in the bottle and the magnitude of the vertical red segment represents the volume of liquid in the bottle at a certain moment of time, and that the resulting trace is a product of tracking how these two quantities covary with respect to conceptual time. That is, the student understands the graph's emergence as being produced by the multiplicative object $(h(t_e), v(t_e))$ for all values t.

The parametric nature of covariational reasoning is apparent in Thompson and colleagues' characterizations (Moore & Thompson 2015; Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2016); a student imagines two quantities varying with respect to conceptual time, thus re-presenting and coordinating these two quantities in a way that variation in either quantity necessarily entails variation in the other quantity. However, the parametric nature of covariational reasoning does not imply that a student reasoning covariationally is consciously aware of a parameter quantity or attribute. For the purposes of this paper, we claim that a student is conscious of the parametric nature of her reasoning when the student indicates awareness of two quantities as simultaneously changing with respect to a third quantity or attribute that is potentially quantifiable. As we illustrate with our data, we do not restrict the parameter quantity or attribute to conceptual time; a student can conceive of two quantities as simultaneously changing with respect to a third quantity or attribute that is not conceptual time, with all three quantities or attributes varying with respect to conceptual time.

2.3. *Carlson et al.'s (2002) characterization of covariation*

In characterizing students' covariational reasoning as described above, we leverage Carlson et al.'s (2002) framework that allows for a fine-grained analysis of students' covariational reasoning. Building on prior researchers' work on covariational reasoning, the authors identified mental actions students engage in when coordinating covarying quantities' magnitudes or numerical values. The mental actions include coordinating *changes in two variables* (quantity A changes as quantity B changes; MA1), *direction of change* (quantity A *increases* as quantity B *increases*; MA2), *amounts of change* (the *change* in quantity A *decreases* as quantity B *increases in equal successive amounts*; MA3), and *rates of change* (quantity A *increases at a decreasing rate* with respect to quantity B; MA4-5) (Fig. 3a).

Carlson et al. (2002) described the aforementioned mental actions in relation to the Bottle Problem (Fig. 3b). As they described, when coordinating the relationship between volume of water and height of water in the bottle as water is poured into the bottle, a student first conceives that the two quantities are changing (MA1). The student can then coordinate that as volume of water increases, height also increases (MA2). Next, the student can coordinate that for equal changes in volume, represented by each colored cross sectional area in Fig. 3c, successive increases in height decrease until the widest part of the bottle, after which increases in height increase until the neck of the bottle (MA3). As the student re-constructs this relationship, she may coordinate the average (MA4) and instantaneous (MA5) rate of change of liquid volume with respect to liquid height as water is poured into the bottle. In this report, we focus on students' enactment of MA1-3 as part of their conceiving and representing relationships between covarying quantities, including the extent that such reasoning is parametric in nature.

Mental Actions of the Covariation Framework

Mental action	Description of mental action
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function

(a)

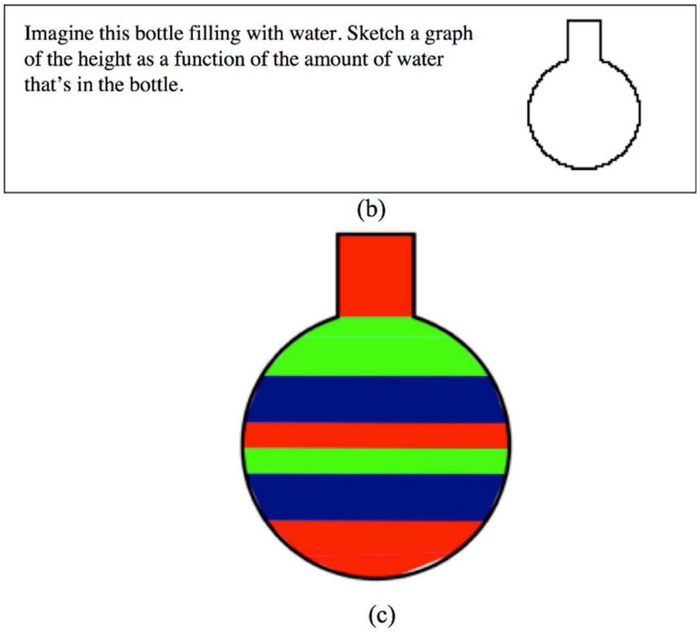


Fig. 3. Carlson et al.'s (2002) framework and the Bottle Problem.

3. Participants and methods

We conducted a teaching experiment with two undergraduate students with the intention of building models of students' mathematics (Steffe & Thompson, 2000). Specifically, we sought to characterize the students' mental actions as they conceived of, reasoned about, and represented relationships between covarying quantities.

3.1. Participants and settings

The participants of this study, 'Arya' and 'Katlyn' were enrolled in a secondary education mathematics program at a large state institution in the southern U.S. Both students were juniors (in credit hours taken) who had successfully completed a calculus sequence and at least two additional courses beyond calculus. The students were enrolled in the first pair of courses (one methods and one content) in the secondary education program. The students were recruited on a volunteer basis from the content course. We chose the students from a pool of volunteers based on their abilities to express their thinking aloud and our judgment that their pre-interview activities were compatible with one another.

We chose to work with pre-service teachers from the aforementioned content course for several reasons. First, the present study is a natural extension of a series of investigations into pre-service teachers' quantitative and covariational reasoning. The content course materials were grounded in this prior research, giving students opportunities to develop meanings for angle measure (Moore, 2013), trigonometric functions (Moore, 2014), and graphing in multiple coordinate systems (Moore, Paoletti, & Musgrave, 2013, 2014). All course materials maintained an explicit focus on students' developing ways of thinking compatible with Carlson, Thompson, and colleagues' descriptions of covariational reasoning, and thus we were interested in studying the students as they progressed in the course.[2]

Second, by focusing on pre-service teachers, we respond to researchers' calls for more attention to the meanings that teachers potentially bring to their classrooms (Simon, 2006; Thompson, 2013). Researchers have identified that teachers' meanings influence their students' meanings, thereby making it critical for mathematics teacher educators to develop models of the meanings pre-service teachers will bring to their classes (Hill, Ball, & Schilling, 2008; Shulman, 1986; Silverman & Thompson, 2008; Stigler & Hiebert, 1999). Such characterizations can help researchers and curriculum designers support pre-service teachers in developing meanings more productive for their teaching.

[2] Although the study was situated in a content course, we are not making any claims in this paper relative to course outcomes or the course causing any of the shifts in student reasoning.

3.2. Data collection and analysis

The present report is situated within a teaching experiment consisting of 16 paired teaching episodes (Steffe & Thompson, 2000) and two semi-structured clinical interviews (Clement, 2000) per student (see Paoletti (2015) for a description of the study in its entirety). The two semi-structured clinical interviews occurred prior to the first teaching episode and after the last teaching episode. Each clinical interview and teaching episode lasted approximately 1.25 hours. We video and audio recorded the sessions and we captured and digitized records of the students' written work at the end of each session.

We used semi-structured clinical interviews (Clement, 2000) as individual pre- and post-interviews (one pre and one post per student) to determine the meanings and reasoning the students brought to the teaching experiment and to examine shifts that we had conjectured occurred throughout the teaching experiment. We used the teaching experiment as an exploratory tool, giving us firsthand experiences with the students' mathematics, and allowing us to explore and promote mathematical progress the students made over the study (Steffe & Thompson, 2000). We analyzed data using both ongoing and retrospective analysis techniques. During each phase of analysis, we conducted conceptual analysis—"building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008). That is, the purpose of conceptual analysis is to develop and refine models of the students' mathematics that viably explain the students' actions.

We note that the research question guiding the overall teaching experiment was: What ways of reasoning do students engage in during activities intended to emphasize reasoning about relationships quantitatively and covariationally? As the teaching experiment progressed, we conjectured the students had developed ways of thinking covariationally that would support them in becoming explicitly aware of the parametric nature of their reasoning. That is, we conjectured the students could reason about two quantities as simultaneously changing with respect to a third quantity or attribute and express awareness of such reasoning (see Section 2.2). We designed the later teaching episodes to test this conjecture, and we report data from the last three teaching episodes due to our explicit focus on the parametric nature of the students' reasoning during our design, enactment, and analysis of those episodes.

3.2.1. Ongoing analysis

Consistent with the teaching experiment methodology (Steffe & Thompson, 2000), we continually built models of the students' mathematics that served as source ideas for designing and adapting tasks for future teaching episodes. Further, we tested these models by predicting how students might respond to a given task or situation. This sometimes led to our refining our initial research question to focus on certain conjectures about students' mathematics created in the moment of interacting with the students or between teaching episodes (Steffe & Thompson, 2000). During our ongoing analysis we debriefed immediately after each episode in order to discuss the students' reasoning and document hypothesized models made in the moment as well as instructional decisions based on these models.

In this report, we describe students' activities resulting from one such instance in which we made a conjecture regarding the students' understandings during our ongoing analysis that we explored in later teaching episodes. Specifically, and as described above, we conjectured that the students' activities led to their creating an intellectual need that resulted in their construction of a parameter quantity or attribute, so we adapted tasks both to test this hypothesis and to explore the extent to which students became explicitly aware of the parametric nature of their reasoning.

3.2.2. Retrospective analysis

We engaged in retrospective analysis after the completion of the teaching experiment; retrospective analysis refers to the activity of reflecting back upon one's hypothetical models with respect to the data collected throughout the teaching experiment (Steffe & Thompson, 2000). Our retrospective analysis involved transcribing the data set and identifying instances offering insights into the students' reasoning. We then performed a conceptual analysis (Thompson, 2008) of these instances, keeping in mind our ongoing analysis efforts, in order to generate and test models of the students' reasoning so that these models provided viable explanations of their behaviors.

With the goal of building viable models of the students' mathematics in mind, we analyzed the records from the teaching episodes using open (generative) and axial (convergent) approaches (Clement, 2000; Strauss & Corbin, 1998). Initially, we identified instances of Arya's and Katlyn's behaviors and actions that provided insights into each student's mathematics. We used these instances to generate tentative models of the students' mathematics that we tested by searching for supporting or contradicting instances in their other activities. When evidence contradicted our constructed models, we made new hypotheses to explain the students' ways of operating and returned to prior data with these new hypotheses in mind for the purpose of modifying previous hypotheses or characterizing shifts in students' ways of operating.

3.3. Task design

Task design is a critical aspect of any teaching experiment (Steffe & Thompson, 2000); a researcher engaging in purposeful task design affords his or her development and testing of models of student reasoning, as well as her or his attempts to engender shifts in student reasoning based on those models. As our initial interest was students' covariational reasoning, throughout the teaching experiment, we provided Arya and Katlyn tasks prompting them to represent relationships between covarying quantities (see Moore, Silverman, Paoletti, & LaForest, 2014 for a detailed description of the tasks used in the content course). For instance, we used a variation of the aforementioned *Bottle Problem*, which is a task initially designed by the Shell Centre (Swan & Shell Centre Team, 1985) and used by several researchers to investigate students' covariational reasoning (e.g., Carlson et al., 2002; Carlson et al., 2001;

Johnson, 2012, 2015; Stalvey & Vidakovic, 2015).

Specific to our version of the bottle problem, we provided the students with a pictured bottle and asked them to imagine the experience of filling the bottle with liquid. Reflecting our interest in students' magnitude reasoning, we did not provide any numerical information on the bottles. We then asked the students to graph the relationship between liquid volume and liquid height in the bottle as it filled with liquid. After they constructed a graph for a given bottle and a bottle for a given graph, we altered the prompt to ask the students to imagine liquid evaporating from the bottle. We then asked the students to represent the relationship between liquid height and liquid volume in the bottle for this new scenario in order to explore how changing situation phenomena may influence the students' understanding of quantities constituting the situation and relationships between those quantities. This variation was based on our prior findings that changing coordinate systems or aspects of a situation can give insights into the extent to which students are reasoning about covarying quantities magnitudes versus engaging in reasoning based in figurative aspects (e.g., how something "looks" or the sensorimotor act of drawing) of a situation or graph (Moore, 2016; Moore & Thompson, 2015; Moore et al., 2013; Moore, Silverman et al., 2014; Moore, Stevens, Paoletti, & Hobson, 2016).

4. Results

We first summarize the students' activities when creating graphs to represent how the liquid height and liquid volume covaried as a bottle filled. We then present their activities addressing liquid evaporating from the bottle in order to illustrate the students representing an additional aspect of the situation in their graph: the direction in which they imagined the graph's emergence, which we interpreted to imply their potentially becoming aware of the parametric nature of their reasoning. We conclude by highlighting the students' activities on a task in the final teaching episode in which the students expressed awareness of the parametric nature of their reasoning.

We note that the students had engaged in constructing and representing relationships between covarying quantities in several contexts and representations (i.e., Cartesian and polar coordinate systems) during the teaching episodes prior to the episodes discussed here. The students exhibited activity consistent with the mental actions described by Carlson et al. (2002) and reasoning about covarying magnitudes as described by Thompson and colleagues (i.e., reasoning about graphs as emergent traces representing a multiplicative object coupling two covarying quantities they conceived as constituting some situation). The students' initial activities addressing the filling Bottle Problem and evaporating Bottle Problem were compatible with such reasoning.

4.1. Overview of students' activities addressing the filling Bottle Problem

During the first part of the Bottle Problem, both students conceived that the two quantities increased in tandem and then determined how the volume of liquid changes for equal successive increases in liquid height; both students coordinated how the volume and height of liquid in a bottle covaried in terms of direction of change (MA2) and amounts of change (MA3). Both students then created a graph while maintaining an explicit focus on how all drawn points and traces represented the relationship she conceived between the height and volume of liquid (i.e., constructed and represented a multiplicative object she understood as invariant between the situation and graph).

As an example that is representative of both students' reasoning, consider Katlyn's activity as she created her graph (see Fig. 4d). Katlyn had already marked equal changes of height on her bottle (Fig. 4a) and shaded the first three areas corresponding to the volume within these height intervals (Fig. 4b) (Excerpt 1).

Excerpt 1. Katlyn coordinates the relationship between liquid volume and liquid height using the bottle [… indicates a break in the transcript].

Katlyn: This volume [*the volume in* (A)] is obviously smaller then this one [*the volume in* (B)]. But they're, like when you add them together like that's the height, or that's the volume at that height. And then when we move from here to here [*pointing to* (B) *then* (C)] um this volume [*the volume in* (C)] is also bigger than the one before it [*the volume in* (B)] so when we add it

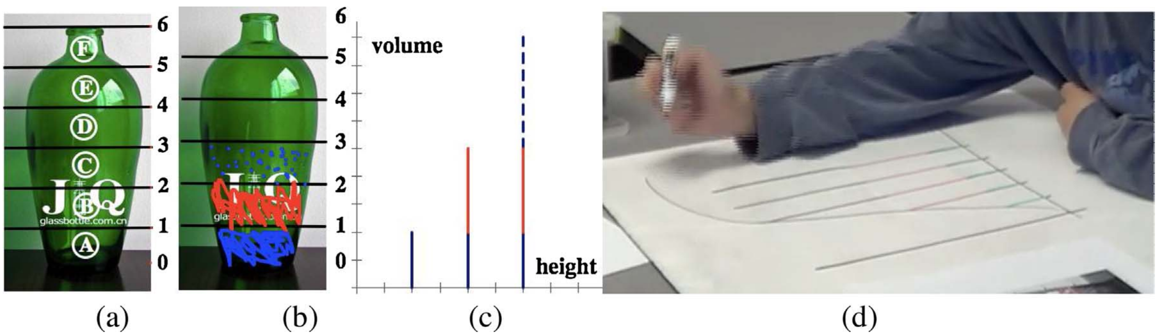


Fig. 4. (a) Katlyn's bottle (numbers and letters added for referencing), (b–c) Katlyn representing total volume with respect to height in the situation and graph, and (d) Katlyn's resultant graph.

on again it's like continuing to grow… the volume is increasing [*pause*] more with equal changes in height.

Katlyn's description of the situation entailed MA1-3 relative to the relationship between liquid height and liquid volume as the bottle filled. Katlyn then drew coordinate axes, marked equal changes along the horizontal height axis, and constructed her graph (Excerpt 2),

Excerpt 2. Katlyn represents the relationship between volume and height using a graph.

Katlyn: This first equal change in height [*pointing to* (A) *in* Fig. 4*a*] gives you like this much volume every single time [*draws segments representing the volume in* (A) *at each height increment, recreated as solid vertical blue segments emanating from the horizontal axis in* Fig. 4*c*]. So you're always going to have that 'cause we're adding them up. Um and then the next one is going, like when we move up in height again [*motioning from* (A) *to* (B) *in* Fig. 4*a*] the next volume that you're going to add on is bigger. But since we're adding them we can add that one to the rest of them [*representing the volume in* (B) *by drawing vertical segments emanating from the endpoint of the original vertical blue segments at the second and each successive height, recreated as longer solid vertical red segments in* Fig. 4*c*].

Katlyn continued in this way as she constructed her graph representing the relationship she understood to constitute the situation. She understood the magnitudes of her drawn color-coordinated vertical segments to represent amounts of volume within specific height intervals, conceiving that each added segment corresponding to an amount of volume added to the total volume (MA1-3). Katlyn's careful attention to the quantities and use of magnitudes indicates she maintained a conception of her graph as an emergent representation of two magnitudes varying with respect to conceptual time as described with Fig. 2. We remind the reader that describing Katlyn as reasoning with respect to conceptual time does not imply that she was consciously aware of the parametric nature of her covariational reasoning at this point in the teaching episode.

4.2. Overview of students' activities addressing the evaporating Bottle Problem

After the students had constructed and discussed height-volume graphs for multiple bottles, we asked them to work together to graph the relationship between height and volume of liquid in the bottle in Fig. 4a as the liquid *evaporated*. We requested they graph this relationship on the same board as a graph previously drawn to represent the relationship between height and volume of liquid in the same bottle as the bottle filled. Indicating they did not anticipate that their previous graph could represent the posed relationship, the pair first drew a new set of axes then spent two minutes considering the posed relationship and deciding how to label their axes. After they decided to label the horizontal axis height, as in Katlyn's original graph (Fig. 4d), Arya noted they should start at "full volume, full height." Katlyn then pointed to the top-right point on her original curve (Excerpt 3).

Excerpt 3. Katlyn describes how to represent the relationship between height and volume as water evaporates from the bottle using a graph.

Katlyn: It's going to look backwards… We can literally just travel this way instead [*motioning over the completed prior graph from the top-right most point back to the origin]. [To the teacher-researchers]* We're done, we're just going to travel this way [*again motioning over the original curve from the top-right most point to the origin].*

[*Shortly after this, Katlyn elaborated as to how she conceived the graph as representing water evaporating from the bottle*]

Katlyn: If we're looking at it like equal changes of height… if we start at this, this height [*pointing to the maximum height magnitude on the horizontal axis*] and this volume [*points to the maximum volume magnitude on the vertical axis then to the top-right most point on the graph*], and then, oh some water evaporated, now we're at this new height [*pointing to tick 4 in* Fig. 4b], now we're going to be at this volume [*placing her marker on her curve corresponding to the height and volume at tick 4*]. Okay more water evaporated [*pointing to tick 3 in* Fig. 4*b*] now we're at this height this volume [*placing her marker on her curve corresponding to the height and volume at tick 3 in* Fig. 4*b*]… We don't need a new graph, it's the same.

Katlyn described this relationship by identifying how liquid volume in the bottle *decreased* for equal *decreases* in liquid height (MA1-2) with respect to this new situation. As when addressing water entering the bottle, Katlyn's careful attention to the quantities' magnitudes represented *on the axes* are indicative of emergent shape thinking. Moreover, by reasoning emergently, she came to understand that her previous completed graph represented the appropriate relationship if traced in the direction opposite to the emergent trace corresponding to the bottle being filled.

To investigate if using the same curve for a new context created a perturbation for the students, the first author asked, "Is the situation the same? You're ending up with the same graph." Katlyn responded, "No, I just want to draw little arrows… we're going this way now [*draws an arrow on the curve pointing towards the origin, recreated in* Fig. 5d]." Addressing how the displayed graph represented two different situations, Katlyn determined a way to differentiate between the situations in the displayed graph by adding an arrow to indicate the direction in which the graph is traced out with respect to the second situation.

We interpret Katlyn's action as her parameterizing the relationship represented by the completed graph with respect to conceptual time, thus differentiating how the completed graph can emerge; we took this activity to suggest her potentially becoming consciously aware of the parametric nature of her reasoning, with the parameter quantity being conceptual time. Adopting Thompson's (2011) notation, we can model Katlyn's understanding of the completed graph as composed of points (h, v) representing the appropriate magnitudes of height and volume of liquid in the bottle, regardless if liquid is entering or leaving the bottle. In the first scenario, she understood $(h, v) = (h(t_\varepsilon), v(t_\varepsilon))$ with t representing conceptual time as liquid enters the bottle and, hence, $h(t_\varepsilon)$ and $v(t_\varepsilon)$ increasing as t increases. By adding the arrow to her graph pointing towards the origin, she indicated that in the second scenario her graph traced from the top-right most point to the origin; she understood $(h, v) = (h(\omega_\varepsilon), v(\omega_\varepsilon))$ with ω representing conceptual time as liquid

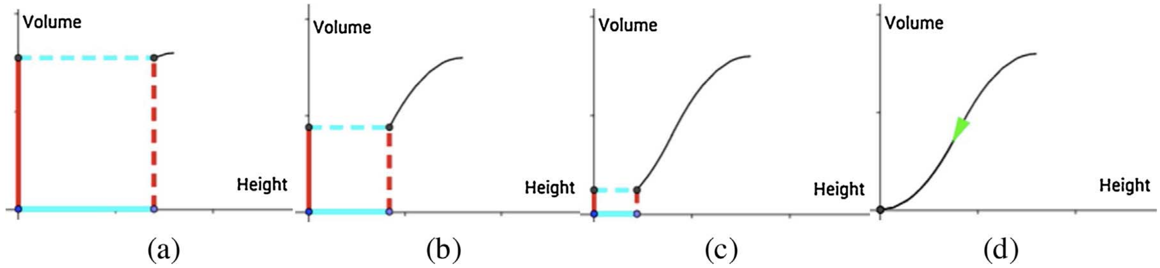


Fig. 5. (a)–(c) A recreation of the students' graph as an emergent trace and (d) a recreation of their graph with the annotated arrow representing the direction of the students' trace.

evaporates from the bottle and, hence, $h(\omega_\varepsilon)$ and $v(\omega_\varepsilon)$ decreasing as ω increases (recreated in Fig. 5a–c).[3]

During our ongoing analysis we noted that this was the first time the students added an arrow to a graph, and we conjectured that the students' actions addressing how the same graph could represent the relationship between volume and height of liquid either as it enters or evaporates from a bottle may have led to their experiencing an intellectual need (Harel, 2007) that resulted in their constructing and becoming conscious of a parameter quantity or attribute. Specifically, although the students conceived a different dynamic trace (i.e., $(h(\omega_\varepsilon), v(\omega_\varepsilon))$ in Fig. 5) than that which produced the initial graph (i.e., $(h(t_\varepsilon), v(t_\varepsilon))$ in Fig. 2), the visual appearance of the two *completed* graphs was the same; the students experienced the intrinsic problem of differentiating between the "same" completed graphs with respect to particular phenomena of the corresponding situations. The students resolved this need by adding an arrow to their graph to indicate the direction of the trace producing the graph, and thus the direction of the variation of each quantity's magnitude in association with the phenomena of the situation.

4.3. Addressing the Car Problem

Consistent with generating and testing hypotheses during ongoing analyses in a teaching experiment, we hypothesized that the students' reasoning had supported them (or could support them) in becoming consciously aware of the parametric nature of their covariational reasoning. We tested this hypothesis during the final teaching episode using an adaptation of the *Car Problem* that Saldanha and Thompson (1998) designed and used to investigate students' covariational reasoning. The task involves an applet showing an individual (Homer) traveling back-and-forth along a road (Fig. 6a). Consistent with their original use, we asked the students to represent the relationship between Homer's distances from two cities (Shelbyville and Springfield) as he travels on the road. After each student constructed a graph representing Homer's distance from each city, we modified the applet to present on the computer screen a normative graphical representation of Homer's distance from each city. Because the relationship is such that neither represented distance is a function of the other distance (i.e., distance from Shelbyville is not a function of distance from Springfield and distance from Springfield is not a function of distance from Shelbyville), we adapted the task by asking about "function" after each student constructed a graph (e.g., "Could you talk about anything in this situation in terms of things being functions?"). We conjectured the students may spontaneously consider other quantities in the situation that were not directly represented in the graph. In turn, the students could have the opportunity to reflect on their reasoning and bring the parametric nature of it to the forefront. We did not expect the students to explicitly define parametric functions, and we were prepared to raise the notion of a parameter quantity of attribute via a parametrically defined function if the students did not do so.

Both students initially described the directional variation (MA2) of each distance (e.g., as Homer moves from the beginning of his trip, the distance from each city decreases). As Arya represented this relationship in her graph, she drew a segment from right to left that corresponded to decreasing ordinate and abscissa magnitudes (indicated by (1) in Fig. 6b). Justifying this segment, Arya pointed to the applet and described, "We start off… far from Springfield and pretty close to Shelbyville [*pointing to Beg. on computer screen then traces along road*]. Then… you're getting closer to Shelbyville for a little ways and closer to Springfield as we're moving along the road" (MA1-2). After describing this relationship using the animation, Arya moved to her graph and marked horizontal dashed lines from each plotted point to the vertical axis to verify that she represented distance from Shelbyville decreasing (indicated by (2) in Fig. 6b). She engaged in compatible activity to verify she represented distance from Springfield was decreasing (indicated by (3) in Fig. 6b). Arya then watched Homer travel past Shelbyville in the animation and stated, "We're moving away from Shelbyville after that and closer to Springfield" (MA1-2). She then represented the relationship she described in the graph by drawing a segment up and to the left (recreated in Fig. 6c) from the left endpoint of the previous graphed segment. Arya continued, "And then, we move away from Springfield again, and away from Shelbyville. And so this, okay. Away from Springfield away from Shelby [*draws segment from left to right, recreated in* Fig. 6d]." (MA1-2)

As in previous situations, Arya conceived her graph as an emergent trace representing a coupling of two covarying magnitudes (i.e., a multiplicative object) as indicated by her explicit attention to the quantities' magnitudes represented along the axes. Further, and similar to the students' actions addressing the Bottle Problem, Arya spontaneously added an arrow to her completed graph

[3] Here we use ω in place of t to indicate that conceptual time, ω, in this case is imagined as varying with respect to the phenomenon of water evaporating from the bottle.

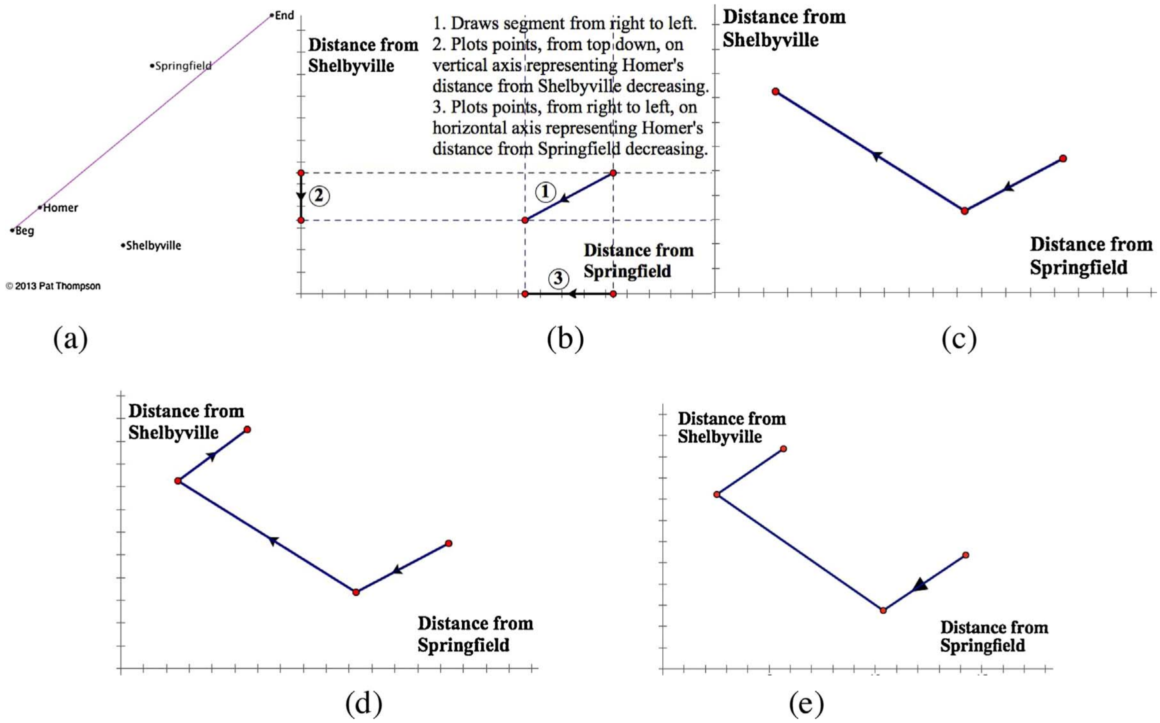


Fig. 6. (a) The Car Problem applet, (b)–(d) a recreation of Arya's work, and (e) a recreation of Arya's final graph.

(Fig. 6e) to represent an additional aspect of the situation: the direction the graph was traced in relation to how Homer traveled along the road from the beginning of his trip. We interpret this to indicate that Arya may have (implicitly or explicitly) had in mind a third quantity or attribute that she was coordinating with the distances from each city.

Intending to explore notion of a parameter quantity of attribute, the first author asked, "Could you talk about anything in this situation in terms of things being functions?" Arya responded by explicitly considering relations with either graphically represented distance as the input and the other distance as the output saying, "If you take either like the distance from Springfield or the distance from Shelby as your input you're going to have more than one output in some places… So neither are really functions." Hoping to raise the idea of a parameter quantity or attribute, the second author asked, "What if your input was total distance traveled and your output was two-dimensional?" He then described the output as being composed of both distance from Springfield and distance from Shelbyville and the following conversation ensued (Excerpt 4).

Excerpt 4. Arya describes how she determines if the relationship with total distance traveled as the input and the two-dimensional output of distance from Springfield and distance from Shelbyville is a function.[4]

Arya: So, but see this is our output right [*motioning over a normative graph displayed on the computer screen*]*?* This [*motioning over the graph*] would be our two-dimensional output. And then [*four second pause*] It's not a function. It says things twice right?
I2: So what are you thinking?
Arya: Wait, that's hard [*nine second pause*]. Yeah [*two second pause*]. Cause there's some ordered pairs we hit twice. Right? No. [*laughs*].
I2: So what are you thinking?
Arya: Function. It's good. [*to self*] Right? Yeah.
I1: Why do you say that?
Arya: 'Cause we don't hit any ordered pairs twice. That's true. Cause if we're traveling then [*motioning over the graph representing distance from Springfield and distance from Shelbyville*] hitting an ordered pair twice would be having an output like, one value hitting the same output, right. So more then one output for an input.
I2: And by input you're thinking of what as the input?
Arya: The total distance. So but that's kinda like [*pointing to graph then shaking her head*], that's not on the graph but whatever.
I2: Yea, so we don't see the input on the graph.
Arya: Right, but these are our outputs [*motioning over the curve*].
[*Shortly after this interaction Arya indicated when describing the relationship as a function she was considering Homer's total distance as being constituted by one trip from Beg to End. I1 asked her if the same relationship would still be a function if she considered accumulated*

[4] I1 refers to the first author and I2 refers to the second author, both of whom acted as teacher-researchers.

total distance as Homer traveled back and forth along the road as the input.]

Arya: That's still, you only have one, cause if that's still, if you're still adding distances going this way [*motioning over the return trip along the road on the computer screen*], these are unique inputs and then they each have just one output. So it's still a function.

Arya initially described that the relation did not represent a function, which we conjectured was because each point on the curve was traced multiple times as Homer traveled back and forth along the road. As she continued, Arya's words and actions indicate that determining if the stated relationship, with a one-dimensional input and two-dimensional output represented a function (e.g., "that's hard" accompanied by several long pauses), was not trivial. We highlight, however, that once she achieved this understanding, she was not perturbed by the fact that the input quantity, total distance traveled, was not expressly represented on her graph (e.g., "that's not on the graph but whatever"). Instead, she conceived that the distances from each city, represented by a coordinate point, simultaneously changed with respect to total distance traveled, and therefore was consciously aware of the parametric nature of her reasoning.

Similarly, Katlyn addressed whether the relationship with the same two-dimensional output but with 'distance on the path' as the input represented a function. She identified, "Well that's what [my graph] shows, right?" and described that for any distance along the path there was only one corresponding coordinate point on her graph that represented Homer's distances from the two cities. Katlyn added, "I understand, like, what I've been drawing this whole time is like, how I'm traveling on like this purple path [*motioning over the road represented on the computer screen*]. But I don't, I never thought of that as my input, but it really is." Like Arya, Katlyn was consciously aware of the parametric nature of her reasoning involving the distances from each city and the parameter quantity distance on the path.

Without prompting, and further suggesting that she was consciously aware of the parametric nature of her reasoning, Katlyn continued to explore and denote the relationship under consideration. Specifically, she wrote "Dist. on Path = > (dist. Shelby, dist. Spring)" as a notational system to consider specific instances of Homer's distance on the path and the corresponding instances of the two-dimensional output. After labeling the point representing Homer's distances from the two cities when Homer was at Beg on the path (labeled A and indicated by (1) for reference in Fig. 7a), Katlyn continued (Excerpt 5).

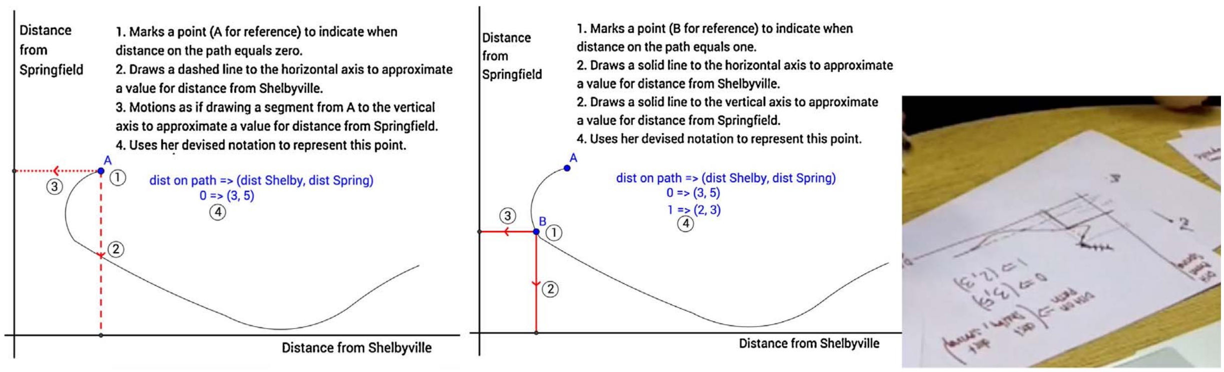
Excerpt 5. Katlyn identifies specific parametrically defined points on her graph using her notation.

Katlyn: So my distance zero on my path [*writes 0 = > , see* Fig. 7*a*], and my output, distance from Shelbyville is [*marking a dashed line from A to the horizontal axis, indicated by (2)*] like, um, three. [*laughing*] Whatever, it doesn't matter, and then the distance from Springfield [*motions as if drawing a line from A to the vertical axis, indicated by (3)*], five [*finishes writing 0 = > (3, 5), indicated by (4)*]. So now I'd say I would start [*pointing to A*] at three five. Then I would do like one [*writes 1 = > as seen in* Fig. 7*b*]. Pretend that this is one [*tracing along her graph from A to B*].

I2: So by, what do you mean by pretending this is one?

Katlyn: [*pointing to B in* Fig. 7*b*] I guess like what, this is like [*pause, points to a position on the road on the computer screen*] one. I don't know, the first portion that I've decided to call one on this purple curve [*using her fingers to indicate an interval starting at Beg along the road in* Fig. 6*a*]. This is [*pointing to B in* Fig. 7*b*] distance on my path, I've traveled a distance of one on my path then I'm this far away from Shelbyville [*drawing a solid segment from B to the horizontal axis, indicated by (2) in* Fig. 7*b*] which is probably like two and then I'm [*drawing solid line from B to the vertical axis, indicated by (3) in* Fig. 7*b*] this far away from Springfield which is like [*finishes writing 1 = > (2, 3), indicated by (4)*]. And so it makes sense but I'm just like, you'd have to go at like really really really really small increments to get like the right thing, you know?

As Katlyn conceived of and represented the relationship with a one-dimensional input and two-dimensional output, she created her own notational system to represent such a relationship. Further, she considered that creating an accurate graph using a parametrically defined function would require "really really really really small" incremental changes in her input or parameter quantity to produce an accurate graph representing the two-dimensional output. We note that although she chose specific values for each



(a) (b) (c)

Fig. 7. (a) and (b) recreations of Katlyn's work and (c) an image of her final work.

quantity (e.g., 0 = > (3, 5)), to Katlyn the actual values were not critical to her reasoning (e.g., "distance from Shelbyville is like, um, three. [*laughing*] Whatever, it doesn't matter"); her reasoning was focused on comparing the magnitudes of the distances from each city represented along the axes with respect to successive distances along the path and she used fictitious numbers to support her in doing this. Throughout, Katlyn's understanding of the graph as an emergent trace, with a trace point simultaneously representing the distance from each city (i.e., a multiplicative object) and corresponding to a particular distance along the path, illustrates her reasoning parametrically and being consciously aware of this reasoning.

5. Discussion and implications

Above, we provided and described empirical examples of students reasoning parametrically and becoming consciously aware of the parametric nature of their reasoning. In this section, we first describe implications of the students' reasoning including connections with relevant literature. We then describe how the students reasoned about one graph as representing multiple possible traces. We conclude with how this reasoning created an intellectual need for parameter quantities or attributes that could be leveraged to develop more formal parametric function understandings.

5.1. Students' covariational reasoning from different lenses

We extend previous research on students' covariational reasoning in several ways. First, Arya and Katlyn's activities illustrate the relation between Keene's definition of dynamic reasoning (i.e., "developing and using conceptualizations about time as a dynamic parameter that implicitly or explicitly coordinates with other quantities" (2007, p. 231)) and Thompson and Carlson, 2017 characterization of conceptual time and covariation. Specifically, Arya and Katlyn's initial activities in each problem were compatible with Keene's description of *implicitly* coordinating time with other quantities and with Thompson and Carlson's description of students maintaining images of varying quantities that tacitly include time. The students then developed images of quantities changing with respect to conceptual time (Thompson & Carlson, 2017), which we infer is necessary for them to engage in what Keene describes as *explicit* dynamic reasoning. We further note that although we would characterize the students' reasoning as dynamic in the *Car Problem,* the students reasoned about distance on the path or total distanced traveled as the parameter quantity, with the parameter quantity and the distances from each city varying with respect to conceptual time.

As we described above, researchers (Carlson et al., 2002; Confrey & Smith, 1994, 1995; Moore, 2016; Moore & Thompson, 2015; Saldanha & Thompson, 1998; Thompson, 2011) have characterized students' covariational reasoning in different ways and with different emphases. Our analysis illustrates that although each characterization of covariational reasoning has a different focus, the characterizations can provide analytic tools that work in tandem. For instance, we illustrated how the students engaging in the mental actions described by Carlson et al. (2002) when conceiving graphs emergently supported them in constructing, reasoning about, and representing relationships between covarying quantities as described by Thompson and Saldanha (Saldanha & Thompson, 1998; Thompson, 2011). Specifically, in order to conceive of relationships in situations and construct accurate graphical representations, the students often enacted MA1-3 (Carlson et al., 2002) whilst reasoning about quantities' magnitudes (Thompson, 2011) and anticipating that their graphs represented emergent traces of a multiplicative object (Moore & Thompson, 2015; Thompson et al., in press).

Returning to the work of Confrey and Smith (1994, 1995), Katlyn introduced specific numerical values when conceiving of the parametrically defined function in the *Car Problem*, adopting a notation reminiscent of a table of values used by Confrey and Smith (1994, 1995). Although numerical values were not critical for Katlyn's reasoning relative to the posed problem, they did support her in considering the need to take smaller and smaller intervals of the parameter in order to convey a method for constructing an accurate graphical representation of a parametrically defined function. Further, we hypothesize that Katlyn's use of numerical values could have become foundational for engaging in reasoning described by Confrey and Smith (1991, 1994, 1995). For instance, if we asked Katlyn to determine average rate of change values relating various quantities in the situation, she could have used her specified values to calculate and coordinate such values. In conclusion, although researchers' descriptions of covariational reasoning can vary in their emphases, each form of reasoning we reference here provides students with useful tools to support their reasoning when addressing different tasks or prompts.

5.2. Covariational reasoning, "the same graph", and implications for learning parametric functions

Moore et al. (2013) described how students' covariational reasoning supported them in conceiving graphs in the Cartesian coordinate system and polar coordinate system as representing invariant relationships. In their case, we interpret the authors to have described students reasoning emergently to conceive a graph in a Cartesian coordinate system representing the same *covariational relationship* as a graph in the polar coordinate system despite perceptual differences in the curves. In our case, we describe students reasoning emergently to understand the same perceptual curve in multiple ways. That is, by reasoning emergently, a student considers the direction in which a graph is traced and thus can understand that a completed graph is *producible by (at least) two different traces representing covarying magnitudes*, with each trace entailing the same coordinate points and thus resulting in a perceptually equivalent curve. As an example, consider two parametric functions, $t \to (x, y)$ and $t \to (u, v)$, $0 \le t \le 2\pi$, such that $(x, y) = (\mathrm{t}, \sin(t))$ and $(u, v) = (2\pi - t, \sin(2\pi - t))$. A student who engages in reasoning compatible with Thompson's and Salandha's descriptions of covariation (Moore & Thompson, 2015; Saldanha & Thompson, 1998; Thompson, 2011) imagines (x, y) and (u, v) as producing different emergent traces (see Fig. 8a–c for the traces produced by (x, y) and (u, v) at three different t values). These traces result in the

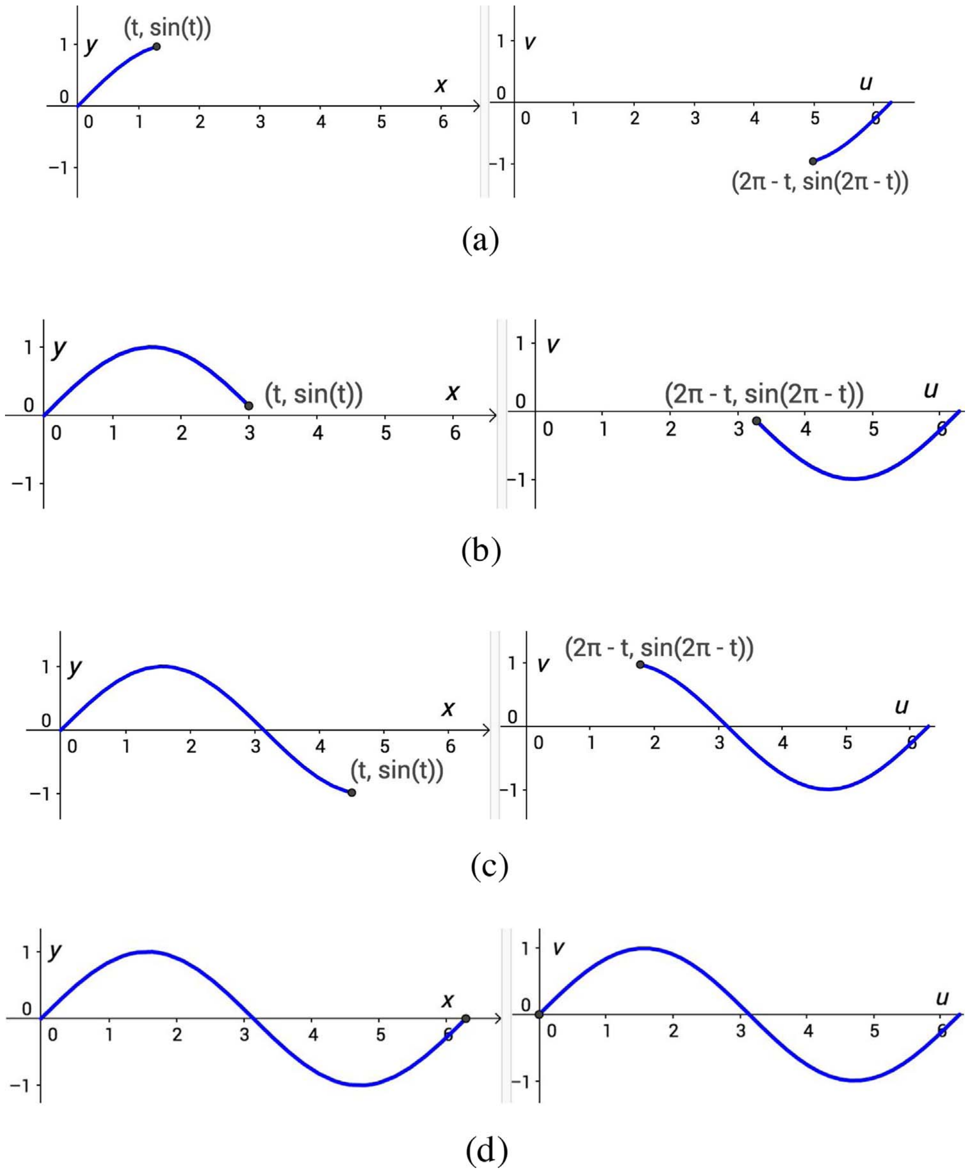


Fig. 8. Four instantiations of the trace of (x, y) = (t, sin(t)) and (u, v) = (2π − t, sin(2π − t)).

same completed graph and, hence, coordinate pairs and perceptual curve (Fig. 8d).

As evidenced by Arya and Katlyn's activities, a student thinking of a completed graph as stemming from multiple emergent traces can experience an intellectual need (Harel, 2007) that results in her explicitly parameterizing a relationship. Because we became aware of such a possibility towards the end of the teaching experiment, we did not have the opportunity to examine the extent to which the students could respond to this need and develop more formal parametric function understandings. However, we note after becoming explicitly aware of the parametric nature of their reasoning, each student was able to interpret a graphical representation in which the parametric variable was not represented. We interpret Arya and Katlyn's actions to be compatible with the interparametric stage described by Trigueros (2001, 2004), which is a more sophisticated level of understanding than that of a majority of students Trigueros studied. Further, we highlight that Katlyn's reasoning included considering the need to take small increments of the parameter in order to construct an accurate graph; such an understanding is critical for developing sophisticated parametric function understandings.

Collectively, the aforementioned findings highlight a potential learning trajectory for parametric functions that rests on students first experiencing and resolving an intellectual need when reasoning about and representing covarying quantities. Specifically, initial notions of parametric functions can stem from students intuitively orienting the direction a completed graph is traced as a means of distinguishing between two real-world scenarios. The students could then identify a third quantity (parameter) on which this distinction is based, and explore the relationship between the original two quantities with this parameter to formalize a parametric function.

6. Closing and looking forward

In this paper, we presented two students' activities that indicated their becoming explicitly aware of the parametric nature of their reasoning when constructing and representing relationships between covarying quantities. We add to the literature on students' covariational reasoning by providing empirical examples of students engaging in reasoning compatible with Thompson and colleagues' (Moore & Thompson, 2015; Thompson, 2011; Saldanha & Thompson, 1998; Thompson & Carlson, 2017) characterizations of covariation and emergent shape thinking, and how this reasoning relates to other characterizations of students' covariational reasoning (Carlson et al., 2002; Confrey & Smith, 1994, 1995). We also illustrated how the students becoming explicitly aware of the parametric nature of their reasoning coincided with their conceiving of quantities as varying with respect to conceptual time and differentiating between experientially different situations. Finally, we add to the literature examining students' parametric reasoning by identifying how particular ways of reasoning can create an intellectual need for parametric functions.

Because we only worked with two students from a particular population, we do not make any claims about the generalizability of our models of the students' reasoning. Instead, we intend that our models provide researchers and educators initial conceptual tools for their work. Future researchers and educators might examine how affording students repeated and sustained opportunities to reason emergently can provide foundations for more explicit and formal introductions to parametric functions that simultaneously focus on students' intellectual need and the mathematical integrity of the content (Harel, 2007). Specifically, researchers could further explore how using different situations that result in students constructing and reasoning about the same completed graph via different emergent traces has the potential to create an intellectual need for parametric functions. As an example, researchers could use similar experiential situations that result in opposite directions of trace such as a Ferris Wheel moving in different directions or a car traveling back and forth from school. Such investigations can leverage both our findings and the genetic decomposition of Stalvey and Vidakovic (2015) to develop more robust ways of supporting students' parametric reasoning and their construction of parametric functions. Furthermore, researchers following such a line of inquiry will contribute more nuanced models of students' mathematics that provide explanatory characterizations of how students' covariational reasoning contributes to their mathematical development.

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