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QUANTITATIVE MEANINGS FOR RELATIONSHIPS: STATES, TRANSFORMATIONS, AND GROSS/QUANTIFIED COVARIATION

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Research on students' and teachers' quantitative reasoning continues to underscore its importance for their learning and development. This importance requires that researchers continue to make strides in identifying salient and important ways of reasoning quantitatively. In this paper, we delineate four forms of quantitative reasoning to characterize students' images of situations. Specifically, we differentiate between students conceiving quantities' changes via state reasoning, transformational reasoning, and gross or quantified covariational reasoning.

Keywords: Mathematical Representations, Cognition, Precalculus

Quantitative reasoning involves conceiving a situation so that it entails measurable attributes (i.e., quantities) and relationships between those attributes (i.e., quantitative relationships; (Smith & Thompson, 2007; Thompson, 2011). *Covariational reasoning* is a form of quantitative reasoning that involves conceiving the ways in which quantities vary in tandem (i.e., covary; Carlson et al., 2002; Confrey & Smith, 1995; Saldanha & Thompson, 1998). Together, quantitative and covariational reasoning (QCR) form a critical foundation for student development at all grade levels (e.g., Ellis, 2011; Johnson, 2015a; Steffe & Olive, 2010; Thompson, 1994). Over the past two decades, we have engaged in research to build models of students' and teachers' QCR. In this theoretical report, we describe four forms of QCR that have emerged as salient during this work: state, transformational, gross, and quantified. We illustrate each form in the context of conceiving of a situation in terms of quantitative relationships.

Informing Covariation Frameworks

Carlson et al.'s (2002) and Thompson and Carlson's (2017) frameworks are two of the most used in the field. Researchers have clarified nuanced, fine-grained ways of reasoning that extend or build on these frameworks (e.g., Ellis et al., 2020; Johnson, 2015a, 2015b; Yu, 2024). We have drawn significantly on these frameworks in our work building models of students' mathematics. Carlson et al.'s (2002) attention to direction and amounts of change have been a foundation for our work in understanding students' QCR (e.g., Liang & Moore, 2021; Moore, 2014; Paoletti et al., 2024). Thompson and Carlson's (2017) levels have aided us in describing how students develop meanings for linear and non-linear relationships (Paoletti & Vishnubhotla, 2022). In our use of covariation frameworks to describe participants' QCR, we identified a need to incorporate a more intentional focus on magnitude reasoning, on non-variational forms of reasoning, and on the extent to which amounts of change are coordinated and compared.

Four Forms of Reasoning: States, Transformations, and Gross/Quantified Covariation

In Table 1, we delineate four forms of reasoning to discuss QCR: (1) state reasoning; (2) transformational reasoning; (3) gross (covariational) reasoning; and (4) quantified (covariational) reasoning. Each form is quantitative in that each involves understanding that two quantities exist in a paired, possibly deterministic relationship. Each also involves understanding that each quantity can take on a multitude of magnitudes, and that changing the magnitude of one quantity might involve changing the magnitude of the other quantity.

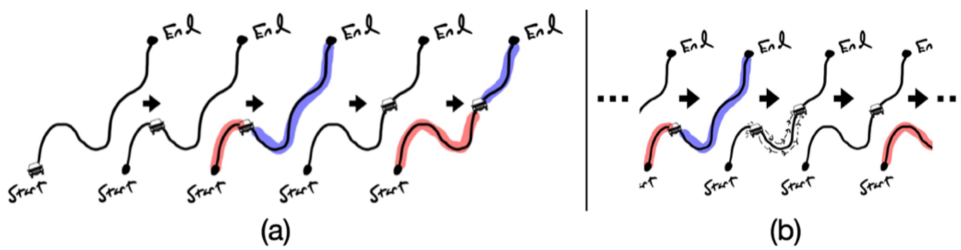
Table 1: Four Forms of Reasoning about Two (or more) Quantities

Reasoning	Description
State	Quantities' paired magnitudes are only dependent on the state under consideration. Changing from one pair to another involves imagining a different instantiation of the situation.
Transformational	Quantities' paired magnitudes are only dependent on the state under consideration. Changing from one pair to another involves transforming between instantiations and reconstructing the magnitudes.
Gross (Covariational)	Quantities' paired magnitudes are dependent on the state under consideration. Additionally, one pair can be changed to another by imagining a quantity's magnitude increasing/decreasing with coordinating increases/decreases in the other quantity's magnitude.
Quantified (Covariational)	Quantities' paired magnitudes are dependent on the state under consideration. Additionally, one pair can be changed to another by constraining their gross covariation by an invariant property of their simultaneous changes.

A person engaged in *state reasoning* conceives the relevant quantities as occurring in distinct instantiations called states. The quantities' magnitudes can change in the sense that there can be different magnitudes at distinct states. The magnitudes at a state exist independently from those at other states. In the context of imagining a road trip, any moment in the trip could be considered a state. At every state, there exists some distance from the start and some distance from the destination. Because each moment exists independent of all others, conceiving another state involves switching attention from one state to the next state *and then* constructing and determining anew the two distance magnitudes (Figure 1a; the red magnitude represents the distance from the start, and the blue magnitude represents distance from the destination). The shift in states produces distinct distance magnitudes, which may or may not have different sizes than in the previous state. For example, a person engaged in state reasoning might focus on the halfway point in their journey and identify that the two distances are equal. They might then shift their focus to a potential rest stop and, with their attention shifted, conceive how far the resting place is from the start and how much distance they have remaining to travel. An individual engaged in state reasoning understands that the path constrains each state. They also might determine the two distances sum to a constant magnitude at every moment in the trip, conceiving this as an invariant property defining the quantities' deterministic relationship at any state.

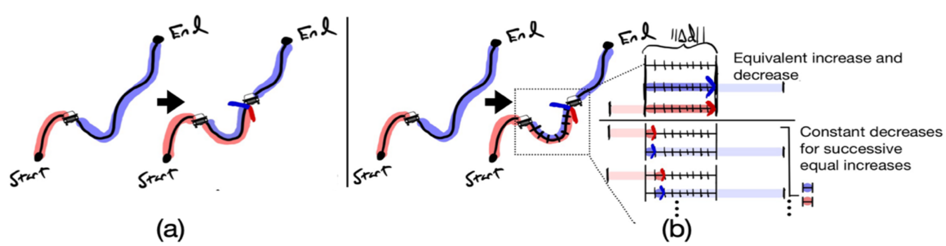
Transformational reasoning is equivalent to state reasoning, but the person imagines transforming the context from one state to produce another state. With a road trip, this involves imagining the car traveling along the road from one state to another, whether smoothly or in chunks (see Figure 1b, which is Figure 1a with the additional conception of the car traveling from one state to another). Like state reasoning, an image of the quantities' magnitudes must be constructed anew at the ending state due to the magnitudes not being sustained while conceiving the transformation. The states are no longer independent of each other, but the quantities' magnitudes constituting each state remain independent from those at other states. Both state and transformational reasoning allow for the quantities' magnitudes to change via being different at different states. Each includes understanding the car's physical position can change, but neither includes images of the (co)variation of two quantities magnitudes. For this reason, we find it productive to characterize such actions as indicating *non-variational quantitative meanings*.

Figure 1: State (a) and Transformational (b) Reasoning About a Road Trip and Distances



A person engaged in *gross (covariational) reasoning* also understands quantities occur in distinct states. That person conceives states with respect to each other; any state is a *snapshot* occurring within the transformational image. Gross covariation additionally involves the quantities' magnitudes being sustained during a transformation. Whether through chunky or smooth reasoning, a person conceives one state of the quantities' magnitudes as dependent on the other state through a process of covariation; the ending state can be produced through a process of covariation emanating from the beginning state. In the case of the road trip, this might involve an individual reasoning that as they proceed from an early rest stop to some state later in the trip, their distance from the start increases and the distance they have remaining decreases (Figure 2a, indicated by the lengthening red magnitude and the shortening blue magnitude). They could conceive this as a loose process of simultaneous increase and decrease, or they could conceive the simultaneous increase and decrease in terms of the specific magnitude each quantity changes in total (e.g., explicitly identifying the magnitude increase in red and the magnitude decrease in blue). As with state and transformational reasoning, they might hold in mind that at any state during that covariation the two distances sum to a constant magnitude.

Figure 2: Gross (a) and Quantified (b) Reasoning About a Road Trip and Two Distances



The aforementioned three forms of reasoning can entail holding in mind an invariant property defining the quantities' overall magnitudes at each state (e.g., they sum to a constant magnitude). A person engaged in *quantified (covariational) reasoning* also constructs a relationship in the *variations of each quantity* so that an invariant property *constrains their covariation* when transforming from one state to another. Gross reasoning dealt with general intervals of increase and decrease and possibly considering specified total change. Quantified reasoning involves more precisely comparing the quantities' variations. In the case of the road trip, this could involve a person reasoning that as they change states, *any increase in the distance* from the start necessitates a decrease of *equal magnitude* in remaining distance (Figure 2b). Or a person might reason that as the distance from the start increases by *successive and equal magnitudes*, the distance remaining decreases by *a constant magnitude* (Figure 2b). In the former, the person quantified the gross covariation by comparing and generalizing one distance's variation relative to the other (i.e., each quantity changes by the same magnitude). In the latter, the person quantified the gross covariation by comparing one quantity's variation across fixed variations in the other (e.g., one quantity's magnitude changes constantly for constant magnitude changes in the other). In each, the quantities' covariation, whether chunky or smooth, was refined through constructing and quantifying amounts of change. Reiterating a point made above, quantified reasoning need not entail reasoning about specific values; in the road trip situation, an individual can coordinate the two distance magnitudes without determining specific values in a unit (Figure 2b). As with the previous three forms of reasoning, the person might also understand that the total distances sum to a constant magnitude at each state. They might also consider how the additional invariant property of covariation maintains the invariant sum property.

A Brief Return to Informing Frameworks

There are numerous connections between these forms and our informing frameworks (Carlson et al., 2002; Thompson & Carlson, 2017). We briefly describe a few motivating needs for formalizing these forms of reasoning. One need we previously alluded to was introducing forms of reasoning that foreground magnitude reasoning. Whereas prior frameworks do not make magnitude reasoning explicit and often use numerical values, describing forms of reasoning with respect to magnitudes is useful and consistent with Thompson et al.'s (2014) emphasis on magnitude-based QCR. Such reasoning is a powerful foundation for students' construction of major mathematical ideas (Liang & Moore, 2021; Liang et al., 2018). As a second need, we have found it useful to make additional distinctions between Carlson et al.'s (2002) directional and amounts of change reasoning. Gross reasoning can entail loose intervals of increase and decrease as well as identifying specified amounts of increase or decrease. The former is consistent with directional reasoning, and the latter is consistent with amounts of change reasoning. Quantified reasoning is also consistent with amounts of change reasoning, but it makes explicit the coordinated comparison of amounts of change. To illustrate, gross reasoning includes understanding that as quantity A increases from 1 to 2 to 3, quantity B increases from 2 to 5 to 9. Quantified reasoning includes additional quantitative operations to compare these increases and conceive quantity B increasing by *increasing amounts* as quantity A increases by *successive constant amounts*. Gross and quantified reasoning underscore important differences in how students can construct and reason about amounts of change. As a third motivating need, Thompson and Carlson (2017) foregrounded images of variation, and specifically differences between chunky and smooth images (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013). Our forms differentiate between reasoning that entails thinking of quantities as merely taking on different amounts (i.e., state and transformational reasoning) and that which entails explicit images of quantities' covariation (i.e., gross and quantified reasoning). Differences between smooth and chunky images are undoubtedly critical and relevant to each form of reasoning we identify, and we thus perceive our forms to complement the distinctions between smooth and chunky reasoning. Because our forms have a slightly different focus than the smooth and chunky distinctions, we have found them to be at times more efficacious in our general work understanding ways to foster and draw on students' or teachers' covariational reasoning.

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